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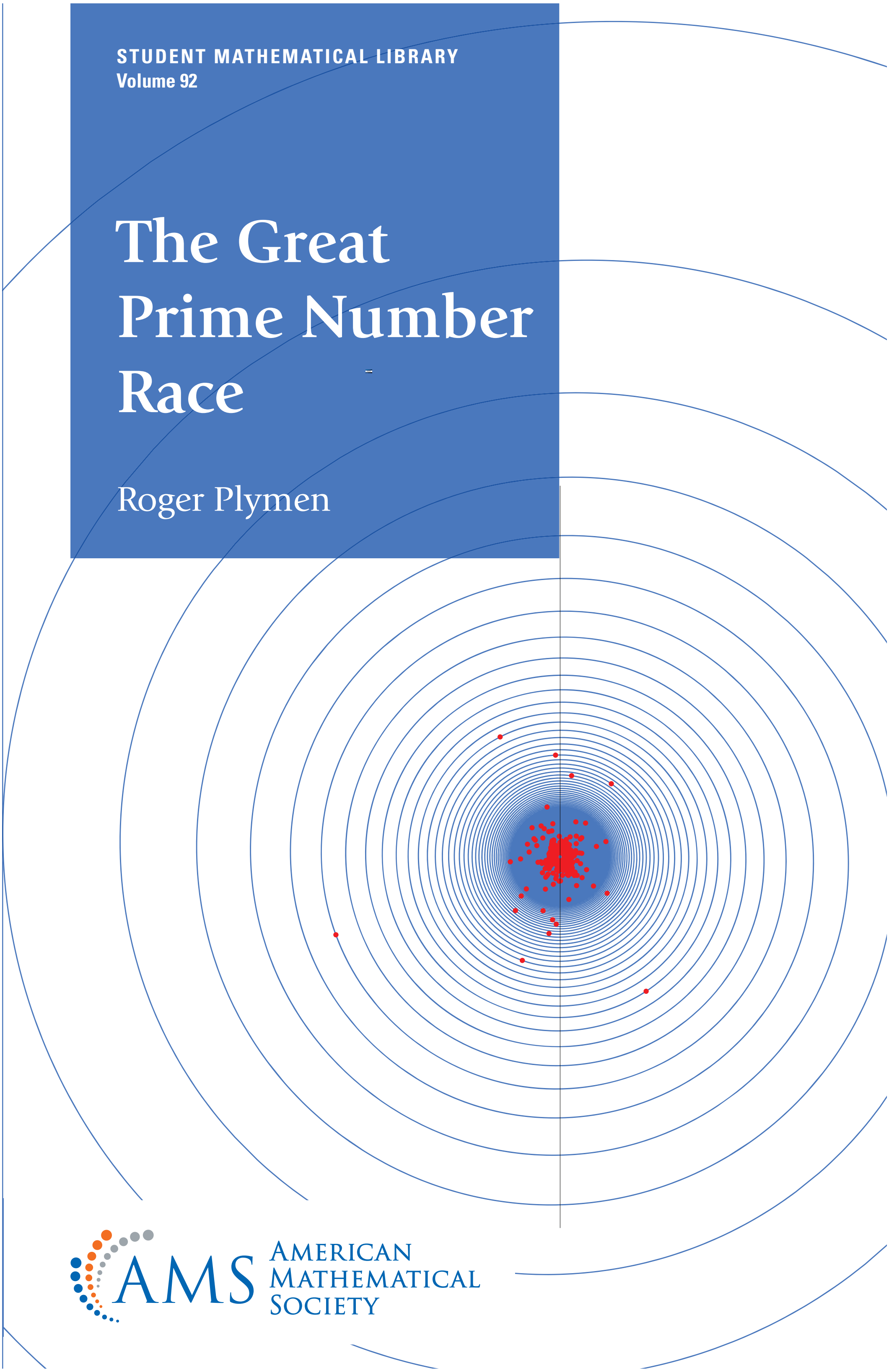
# The Great Prime Number Race

Roger Plymen



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# The Great
Prime Number
Race

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Roger Plymen

AMS American Mathematical Society
Providence, Rhode Island

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To Hilary

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# Preface

This is a book about prime numbers. Let $\pi(x)$ denote the number of primes less than or equal to $x$, and let $\mathrm{Li}(x)$ denote the logarithmic integral of $x$. According to the prime number theorem, we have

$$\pi(x) \sim \mathrm{Li}(x).$$

This is an asymptotic statement: it means that

$$\frac{\pi(x)}{\mathrm{Li}(x)} \to 1 \qquad \text{as} \quad x \to \infty.$$

If you look at any table of primes, you will find that $\pi(x) < \mathrm{Li}(x)$. The numerical evidence for this is quite strong. In fact, we know that

$$\pi(x) < \mathrm{Li}(x)$$

for $2 \leq x \leq 10^{19}$; see [**But**, Theorem 2(1.10)].

Nevertheless, it remains a fact, proven by Littlewood in 1914 [**Li**], that the sign of $\pi(x) - \mathrm{Li}(x)$ oscillates infinitely often!

It seems appropriate to describe this phenomenon as a "race" between $\pi(x)$ and $\mathrm{Li}(x)$. Sometimes $\mathrm{Li}(x)$ is ahead; sometimes $\pi(x)$ is ahead. According to the numerical evidence, $\mathrm{Li}(x)$ starts in front. So, when is the first time that $\pi(x)$ gets ahead? That is to say, what is the smallest natural number for which the prime number theorem *undercounts* the number of primes?

The Skewes number is, by definition, the least natural number with this property. The Skewes number will be denoted $\Xi$. So we have

$$\pi(\Xi) > \mathrm{Li}(\Xi)$$

and $\Xi$ is the least number with this property.

What is $\Xi$ ? No one knows. The best one can do is to provide an upper bound for $\Xi$. The best upper bound for $\Xi$, at the time of this writing, is a number with 317 digits!

I should mention that there is a broad class of prime number races, which are pursued in the recent article by Lichtman, Martin, and Pomerance [**LMP**]. Nevertheless, the race between $\pi(x)$ and $\mathrm{Li}(x)$ holds its position as the most important of these races, hence the title of this book.

The mathematical background for the Skewes number is quite extensive, and the aim of this book is to provide this background. In particular, we provide proofs of the explicit formulas of analytic number theory, a proof of the oscillation theorem of Littlewood, a refinement of Lehman's theorem, and a concise proof of the prime number theorem (due to Newman-Zagier).

The main aspects of the mathematical background are

- the Riemann explicit formula, an equation between the prime counting function on one side, the zeta zeros on the other side,
- Lehman's theorem, an integrated version of the Riemann formula, which is amenable to numerical calculation,
- numerical analysis using a large number of zeta zeros.

For my part, the endless fascination of the Skewes number lies in the interaction between the deep conceptual formulas of Riemann and Lehman and the subsequent numerical analysis using, for example, 22 million zeta zeros.

The Skewes number is named after the South African mathematician Stanley Skewes, who was a research student of John Edensor Littlewood.

I have written this book with an undergraduate audience in mind. If some of the proofs seem hard, then skip them at a first reading. To paraphrase the author of Ecclesiastes: there is a time to read proofs and a time to *refrain* from reading proofs. To leaven the experience of reading the proofs, I have provided some historical context and background. This includes a section on Kontsevich-Zagier periods [**KZ**] and some scholarly historical material.

Overall, this short book will serve as an invitation, a gentle introduction, to the explicit formulas of analytic number theory, using the Skewes number as thread and motivation.

The book is an expansion of lectures which have proven suitable for third-year undergraduates in the UK.

I take a fairly direct pathway to the explicit formulas, and in this context it's a pleasure to acknowledge my debt to Jeffrey Stopple's book [**St**].

Chapter 8 comprises a set of exercises which have been found suitable for an undergraduate course. Hints, and sometimes complete solutions, are also provided.

Throughout this text, the emphasis is on the Riemann explicit formula. My experience is that undergraduates respond well to the splendor of the Riemann explicit formula.

The Riemann explicit formula is an equation of the following form:

$$\begin{aligned}&a\ counting\ function\ depending\ only\ on\ the\ prime\ numbers\\&\quad = a\ formula\ depending\ only\ on\ the\ zeta\ zeros.\end{aligned}$$

Note that, in the explicit formula, the left-hand side depends only on the prime numbers and the right-hand side depends only on the zeta zeros: this is sometimes referred to as *the reciprocity between the primes and the zeta zeros.*

This formula contains oscillatory terms corresponding to the complex zeros of the Riemann zeta function. In the book by Mazur and Stein [**MS**, p. 103], the set of the imaginary parts of the complex zeros of $\zeta(s)$ is called the *Riemann spectrum of the primes.* If each number in the Riemann spectrum is interpreted as a frequency, then the Riemann spectrum is a sound spectrum, i.e., a *chord* in the musical sense, with a definite pitch (the fundamental frequency) and a certain timbre (the overtone series). At this point in the book, there are some original touches: the chord of the Riemann spectrum gets louder as time goes by! There is a sense in which one should be able to hear the Riemann hypothesis.

I have been especially influenced by *Prime obsession*, the excellent popular book by John Derbyshire [**Derb**], by the detailed analysis in Edwards [**Ed**] of the Riemann memoir of 1859, and by Jeffrey Stopple's book [**St**]. We have, above all, been influenced by the Riemann memoir [**R**]. My historical sources are

– the articles by R. Ayoub [**A**], J. C. Lagarias [**L**], T. J. Osler [**Osler**], V. S. Varadarajan [**V**],

– the chapter on Euler in André Weil's book [**We**], Euler's book *Introduction to analysis of the infinite* [**E**], Riemann's original memoir [**R**],

– the Euler archive [**Euler**].

I first met the Skewes number in Littlewood's *Miscellany* [**Li**, pp. 110–112], which I read as a schoolboy. Somehow, the Skewes number lingered at the back of my mind, and, when in 2006–2007, I had the opportunity to work on it with my MSc student Kuok Fai Chao, I took the opportunity. This led to the article [**CP**], the first of recent papers to re-examine Lehman's proof.

It is a pleasure to thank Nick Gresham for help, at various times, with computational aspects. I wish to thank Christine Lee for many insightful discussions. I also wish to thank the third-year undergraduates at Manchester University who attended my course, for their stimulating and varied responses to my lectures. Their responses definitely affected the shape of the lectures, hence the shape of this book.

Finally, it's a pleasure to thank Eriko Hironaka, AMS Editor, for her patient support and encouragement over a period of several years.

Roger Plymen

## Chapter 1

# The Riemann zeta function

### 1.1. Introduction

Let $\pi(x)$ denote the number of primes less than or equal to $x$. We have the logarithmic integral

$$\mathrm{Li}(x) := \int_0^x \frac{dt}{\log t}.$$

This integral requires careful discussion. Owing to the singularity of the function $t \mapsto \frac{1}{\log t}$ at $t = 1$, we have to define this integral via the Cauchy principal value taken at 1:

$$\mathrm{Li}(x) := \lim_{\epsilon \to 0+} \left( \int_0^{1-\epsilon} + \int_{1+\epsilon}^x \right) \frac{dt}{\log t}.$$

The reader will sometimes encounter the function

$$\mathrm{li}(x) := \int_2^x \frac{dt}{\log t}.$$

It is immediate that

$$\mathrm{Li}(x) = \mathrm{Li}(2) + \mathrm{li}(x)$$

for $x \geq 2$. The notation for $\mathrm{Li}(x)$ and $\mathrm{li}(x)$ is not universal: the current notation in the Wikipedia page on the logarithmic integral function is the reverse of the present notation. Here, we are following the notation in [**MV**, p. 189].

According to the prime number theorem, we have

$$\pi(x) \sim \mathrm{Li}(x).$$

This is an asymptotic statement: it means that

$$\frac{\pi(x)}{\mathrm{Li}(x)} \to 1 \qquad \text{as} \quad x \to \infty.$$

It is worth noting, in passing, that $\mathrm{Li}(2) \approx 1.045163\ldots$, so that $\pi(2) < \mathrm{Li}(2)$, albeit by a narrow margin. Indeed, there is a mass of numerical evidence which suggests that $\pi(x) < \mathrm{Li}(x)$. We have the following remarkable result of J. Buthe:

$$\pi(x) < \mathrm{Li}(x)$$

for $2 \leq x \leq 10^{19}$; see [**But**, Theorem 2(1.10)].

Nevertheless, it remains a fact, proven by Littlewood in 1914 [**Li**], that the sign of $\pi(x) - \mathrm{Li}(x)$ oscillates infinitely often!

Consider the set

$$\{x \in \mathbb{N} : \pi(x) > \mathrm{Li}(x)\}.$$

Littlewood's theorem implies that this set has a least element. The least element in this set of natural numbers will be called *the Skewes number.* We will denote the Skewes number by $\Xi$:

$$\Xi := \inf\{x \in \mathbb{N} : \pi(x) > \mathrm{Li}(x)\}.$$

We can be sure that $10^{19}$ is a *lower bound* for the Skewes number $\Xi$:

$$10^{19} < \Xi.$$

What is $\Xi$? No one knows. The best one can do, at the time of writing, is to find an upper bound for $\Xi$. Such an upper bound will be called *a Skewes number.*

In due course, Skewes [**S**] established his famous upper bound:

$$\exp\exp\exp\exp(7.705) < 10^{10^{10^{10^{3}}}}.$$

**Theorem 1.1** (Skewes [**S**]). *For some* $x < \exp\exp\exp\exp(7.705)$ *we have*

$$\pi(x) > \mathrm{Li}(x).$$

In this remarkable article, the proofs are based on both of the explicit formulas: the von Mangoldt formula (see Section 4.1) is quoted in [**S**, Part I, Eq. (2)], and the Riemann formula (see Section 4.6) is quoted in [**S**, Part II, Eq. (24)].

Subsequent developments by Lehman and others all depend crucially on the Riemann explicit formula.

How do we reduce the upper bound? Lehman decided to *integrate* the difference $\pi(x) - \mathrm{Li}(x)$ against a positive test function over a certain interval $[a, b]$. If the resulting integral is positive, then we can infer that $\pi(x) > \mathrm{Li}(x)$ for some $x$ in the interval $[a, b]$. To prove that the Lehman integral is positive, the only known way is to enter millions of zeta zeros with the help of a powerful computer. This method has been successively refined, and the current best known bounds are as follows. We will write them in the natural log scale:

$$19 \log 10 < \log \Xi < 727.951346801.$$

There are some very large natural numbers, for example, the estimated number of stars in the universe, or the estimated number of viruses in the oceans of the world, but these numbers are dwarfed by the upper bound $\exp(727.951346801)$, which is a colossal number with 317 digits.

## 1.2. The Riemann zeta function

In this section, we will meet the Riemann zeta function, named after Bernhard Riemann (1826–1866). The notation $\zeta$ for the function and $s$ for the variable are both due to Riemann in his memoir of 1859.

Euler (1707–1783) had worked extensively on infinite series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^k}$$

where $k$ is a positive integer, see his famous book *Introduction to analysis of the infinite*, Chapter X [**E**]. In Riemann's notation, we would write

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}.$$

In fact, Euler discovered an elegant formula for $\zeta(k)$ where $k$ is an even positive integer.

Riemann's inaugural dissertation of 1851 was called "Foundations for a general theory of functions of a complex variable" . With this in his mathematical background, it was perhaps natural for Riemann to interpolate between the positive integer values of $k$, thereby leading to the zeta function $\zeta(s)$ of a *complex variable* $s$. With hindsight, this may seem like a natural, even inevitable, step, but it is surely a big conceptual leap. When this is done, there are issues of convergence and we have

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

with $\Re(s) > 1$.

We will show in this chapter that the zeta function can be extended to a meromorphic function in the entire plane $\mathbb{C}$. We briefly review some of Euler's pathbreaking work on $\zeta(2k)$.

With acknowledgement to Euler, the zeta function $\zeta(s)$ is sometimes called the Euler-Riemann zeta function.

Riemann's memoir is to be found in [**R**], and English translations are readily available; see for example [**Ed**].

## 1.3. The prime numbers

**Definition 1.2.** An integer $n \geq 2$ is prime when the only divisors of $n$ are 1 and $n$ itself. If $n \geq 2$ is not prime, then it is said to be composite, or nonprime.

We can list the first few prime numbers:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37 \ldots.$$

Then given any positive integer $x \in \mathbb{Z}$ we define $\pi(x)$ to be the number of prime numbers less than or equal to $x$. For example $\pi(10) = 4$, $\pi(20) = 8$, $\pi(40) = 12, \ldots.$

**Theorem 1.3.** *There exist infinitely many prime numbers.*

**Proof.** This is from Euclid's *Elements* IX 20. It is really a one-line proof: let $p_1, p_2, \ldots, p_n$ be the first $n$ prime numbers and consider

$$Q = 1 + p_1 \cdot p_2 \cdots p_n.$$

To spell out the details:

– EITHER $Q$ is prime, in which case there is a prime, namely $Q$, greater than $p_n$

– OR $Q$ is nonprime, in which case each prime factor of $Q$ must be greater than each $p_j$, for division of $Q$ by $p_j$ admits remainder 1.

Either way, there is a prime greater than $p_n$. This proof does not generate primes, but it does place an upper bound, namely $Q$, on the next prime after $p_n$. The first few steps yield

$$\begin{gathered} 1+2=3, \\ 1+2\cdot 3=7, \\ 1+2\cdot 3\cdot 5=31, \\ 1+2\cdot 3\cdot 5\cdot 7=211, \\ 1+2\cdot 3\cdot 5\cdot 7\cdot 11=2311, \\ 1+2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13=30031. \end{gathered}$$

Of these, $3, 7, 31, 211, 2311$ are prime; $30031$ is composite with least prime factor 59. ☐

**Definition 1.4.** A Mersenne prime is a prime number of the form $M_n = 2^n - 1$ for some integer $n$.

They are named after Marin Mersenne, a French Minim friar, who studied them in the early 17th century. As of December 2018, the largest known Mersenne prime is

$$2^{82,589,933} - 1,$$

a number with $24,862,048$ digits.

Note the algebraic identity

$$y^m - 1 = (y-1)(1 + y + y^2 + \cdots + y^{m-1})$$

and set $y = 2^k$. Then we have that $2^k - 1$ divides $2^{km} - 1$, and so if $n \in \mathbb{Z}$ is composite, then $2^n - 1$ is composite. In other words, if $M_n$ is prime, then $n$ must be prime. Each Mersenne prime is therefore of the form

$$M_p = 2^p - 1.$$

For more information on this topic, go to GIMPS, the Great Internet Mersenne Prime Search.

## 1.4. The Riemann zeta function

We begin with the definition of the zeta function. This definition first appeared in the famous Riemann memoir of 1859.

**Definition 1.5.** The Riemann zeta function is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

with $s \in \mathbb{C}$, $\Re(s) > 1$.

Writing $s = \sigma + it$, we obtain

$$\begin{aligned} \frac{1}{n^s} &= n^{-s} \\ &= e^{-s \log n} \\ &= e^{-\sigma \log n - it \log n} \\ &= e^{-\sigma \log n} \cdot e^{-it \log n} \end{aligned}$$

and so

$$\frac{1}{n^s} = n^{-\sigma} \left( \cos\left(t \log n\right) - i \sin\left(t \log n\right)\right).$$

We can now write

$$\Re\left(\zeta(s)\right) = \sum_{n=1}^{\infty} \frac{\cos\left(t \log n\right)}{n^{\sigma}} \quad \text{and} \quad \Im\left(\zeta(s)\right) = -\sum_{n=1}^{\infty} \frac{\sin\left(t \log n\right)}{n^{\sigma}}$$

and also $|1/n^s| = 1/n^{\sigma}$.

It's useful to study the zeta function for some real values of the variable $s$.

**Theorem 1.6.** *The infinite series given by* $\sum_{n=1}^{\infty} \frac{1}{n^s}$ *has the following properties:*

(1) *The series converges if* $s > 1$.

(2) *The series diverges if* $s = 1$.

**Proof.** Suppose first that $s > 1$; then observe that

$$\frac{1}{2^s} + \frac{1}{3^s} < \frac{1}{2^s} + \frac{1}{2^s} < \frac{2}{2^s} < 2^{1-s}$$

and

$$\frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} < \frac{1}{4^s} + \frac{1}{4^s} + \frac{1}{4^s} + \frac{1}{4^s} < \left(2^{1-s}\right)^2$$

and so on. The geometric series

$$f(s) := \sum_{n=0}^{\infty} \left(2^{1-s}\right)^n$$

converges if $s > 1$ for then we have $0 < 2^{1-s} < 1$. This gives us that

$$s > 1 \implies \sum_{n=1}^{\infty} \frac{1}{n^s} < \infty.$$

Next, we consider the case when $s = 1$ and we consider the $n$th harmonic number

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}.$$

For the harmonic numbers we have the following elegant and important inequalities:

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2},$$

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2},$$

$$\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \cdots + \frac{1}{16} > \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \cdots + \frac{1}{16} = \frac{1}{2},$$

and so on. In general, we have

$$H_{2^k} > 1 + k/2$$

and so

$$H_{2^k} \to \infty \quad \text{as} \quad k \to \infty.$$

The conclusion is that the harmonic series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$$

is a *divergent* series. ☐

The divergence is surprisingly slow. Here's a thought experiment to emphasize this point. A computer is programmed to add 1 trillion terms of the harmonic series $\sum \frac{1}{n}$ each second. Since the Big Bang, has enough time elapsed for the $N$th partial sum to exceed 100 ?

Since

$$\left|\frac{1}{n^s}\right| = \frac{1}{n^{\Re(s)}},$$

it is clear that

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

is absolutely convergent in the open half-plane $\Re(s) > 1$.

## 1.5. Euler and the zeta function

Euler found formulas for $\zeta(s)$ when $s$ is an even natural number. In his book *Introduction to analysis of the infinite*, he showed that [**E**, pp. 139–140]

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2^0}{3!}\frac{1}{1}\pi^2,$$

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{2^2}{5!}\frac{1}{3}\pi^4,$$

$$\zeta(6) = \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{2^4}{7!}\frac{1}{3}\pi^6,$$

$$\zeta(8) = \sum_{n=1}^{\infty} \frac{1}{n^8} = \frac{2^6}{9!}\frac{3}{5}\pi^8,$$

$$\zeta(10) = \sum_{n=1}^{\infty} \frac{1}{n^{10}} = \frac{2^8}{11!}\frac{5}{3}\pi^{10},$$

$$\zeta(12) = \sum_{n=1}^{\infty} \frac{1}{n^{12}} = \frac{2^{10}}{13!}\frac{691}{105}\pi^{12},$$

$$\zeta(14) = \sum_{n=1}^{\infty} \frac{1}{n^{14}} = \frac{2^{12}}{15!} \frac{35}{1} \pi^{14},$$

$$\zeta(16) = \sum_{n=1}^{\infty} \frac{1}{n^{16}} = \frac{2^{14}}{17!} \frac{3617}{15} \pi^{16},$$

$$\zeta(18) = \sum_{n=1}^{\infty} \frac{1}{n^{18}} = \frac{2^{16}}{19!} \frac{43867}{21} \pi^{18},$$

$$\zeta(20) = \sum_{n=1}^{\infty} \frac{1}{n^{20}} = \frac{2^{18}}{21!} \frac{122277}{55} \pi^{20},$$

$$\zeta(22) = \sum_{n=1}^{\infty} \frac{1}{n^{22}} = \frac{2^{20}}{23!} \frac{854513}{3} \pi^{22},$$

$$\zeta(24) = \sum_{n=1}^{\infty} \frac{1}{n^{24}} = \frac{2^{22}}{25!} \frac{1181820455}{273} \pi^{24},$$

$$\zeta(26) = \sum_{n=1}^{\infty} \frac{1}{n^{26}} = \frac{2^{24}}{27!} \frac{76977927}{1} \pi^{26}.$$

For the original document, see [**Euler**, E130|1739, §24]. Returning to his book, Euler writes [**E**, p. 140]:

*...we have gone far enough to see a sequence which at first seems quite irregular,* $1, 1/3, 1/3, 3/5, 5/3, 691/105, 35/1, \ldots$*, but it is of extraordinary usefulness in several places.*

Later, Euler discovered the pattern underlying these apparently irregular numbers and obtained his beautiful formula

$$\zeta(2k) = \frac{(-1)^{k+1} 2^{2k-1} B_{2k}}{(2k)!} \pi^{2k}.$$

Using $\eta(s) = (1 - 2^{1-s})\zeta(s)$ we have

$$\eta(2k) = \sum_{n=1}^{\infty} \frac{(-1)^{k+1}}{n^{2k}} = \frac{(-1)^{k+1}(2^{2k-1} - 1) B_{2k}}{(2k)!} \pi^{2k}.$$

Here the numbers $B_k$ are the Bernoulli numbers, and they are all rational. For the first few we have $B_0 = 1, B_1 = -1/2$, and

$$B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42},\ B_8 = \frac{1}{30},\ B_{10} = \frac{5}{66},\ B_{12} = -\frac{691}{2730},$$
$$B_3 = B_5 = B_7 = \cdots = 0$$

and their generating function is

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.$$

Notice that $\zeta(s)$, with $s$ an even natural number, is a transcendental number.

## 1.6. Meromorphic continuation of $\zeta(s)$

We say that a function $f(s)$ with $s \in \mathbb{C}$ is *holomorphic* in an open set $U \subseteq \mathbb{C}$ if the derivative $df/ds$ exists for all $s \in U$. We write $\mathrm{Hol}(U)$ for the set of all complex functions that are holomorphic in the open set $U$. The set $\mathrm{Hol}(U)$ is a vector space: for any $f, g \in \mathrm{Hol}(U)$ and $\lambda \in \mathbb{C}$, we have

$$\frac{d(f+g)}{ds} = \frac{df}{ds} + \frac{dg}{ds},$$
$$\frac{d(\lambda f)}{ds} = \lambda \left( \frac{df}{ds} \right).$$

The kind of convergence which is useful in complex analysis is rather strict: we require uniform convergence on compact sets. Uniform convergence on compact sets is called *locally uniform convergence*. The usefulness of this kind of convergence is spelled out in the following important lemma.

**Lemma 1.7 (The convergence lemma).** *Let $U \subseteq \mathbb{C}$ be an open set, and let $f_n \in \mathrm{Hol}(U)$ be a sequence of functions. If $f_n \longrightarrow f$ locally uniformly, then $f \in \mathrm{Hol}(U)$ and*

$$\frac{df_n}{ds} \longrightarrow \frac{df}{ds} \qquad \text{as} \quad n \longrightarrow \infty$$

*also locally uniformly.*

We continue with an immediate application of the convergence lemma.

**Theorem 1.8.** *The Riemann zeta function $\zeta(s)$ is holomorphic in the open half-plane $\Re(s) > 1$.*

**Proof.** Let $\mathbf{D} = \{s \in \mathbb{C} : \Re(s) > 1\}$. Note that for any $\delta > 0$, for all $s \in \mathbf{D}$ such that $\Re(s) \geq 1 + \delta$, we have

$$\sum \left| \frac{1}{n^s} \right| \leq \sum \frac{1}{n^{1+\delta}} < \infty.$$

In other words, since for each $n$ the function $1/n^s$ is holomorphic everywhere and the sum converges uniformly on compact sets of $\mathbf{D}$, $\zeta$ is holomorphic on $\mathbf{D}$ by the convergence lemma. □

Before discussing the meromorphic continuation of the zeta function, we recall the four types of isolated singularities: removable singularity, simple pole, pole of order $n$, essential singularity. We illustrate via singularities at 0.

(1) **Removable singularity at** $0$**.**

$$\begin{aligned} \frac{\sin z}{z} &= \frac{1}{z} \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \\ &= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots . \end{aligned}$$

The apparent singularity at $z = 0$ can be removed.

(2) **Simple pole at** $0$**.**

$$\begin{aligned} \frac{\sin z}{z^2} &= \frac{1}{z^2} \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \\ &= \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \frac{z^5}{7!} + \cdots . \end{aligned}$$

The $1/z$ term gives rise to the simple pole at $z = 0$.

(3) **Pole of order** $n$ **at** $0$. An example of a pole of order $n$ is an expression of the form $1/z^n$. This is a pole of order $n$ at $z = 0$. Note that a simple pole is a pole of order 1.

(4) **Essential singularity at** 0. Here is a good example.

$$\begin{aligned} e^{1/z} &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n \\ &= \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} \\ &= 1 + \frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z^2}\right) + \frac{1}{3!}\left(\frac{1}{z^3}\right) + \cdots . \end{aligned}$$

**Definition 1.9.** A function $f(z)$ of a complex variable $z \in \mathbb{C}$ is meromorphic in an open set $D \subseteq \mathbb{C}$ if it is holomorphic at all points except for isolated poles. The function $f(z)$ is entire if it is holomorphic at all points $z \in \mathbb{C}$.

Now consider the power series

$$f(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

provided that $|z| < 1$. The function $f(z) = 1/(1-z)$ is defined everywhere in $\mathbb{C} - \{1\}$ even though the series $\sum_{n=0}^{\infty} z^n$ only converges in the open disc $|z| < 1$.

Note that $f(z) = 1/(1-z)$ is not defined at $z = 1$ as there is a simple pole there. But by Definition (1.9), this function is meromorphic as the pole is an isolated singularity. This idea is called meromorphic continuation, which we now illustrate with the Riemann zeta function.

**Theorem 1.10.** *The meromorphic continuation of the Riemann zeta function* $\zeta(s)$ *into the open half-plane* $\Re(s) > 0$ *is given by*

$$\zeta(s) = 1 + \frac{1}{s-1} - s\int_1^{\infty} \frac{\{x\}}{x^{s+1}} dx. \tag{1.1}$$

**Proof.** For $\Re(s) > 1$ we have

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^{\infty} n^{-s}\\ &= \sum_{n=1}^{\infty} s\int_n^{\infty} \frac{dt}{t^{s+1}}\\ &= s\int_1^{\infty}\left(\sum_{n\le t} 1\right)\frac{dt}{t^{s+1}}\\ &= s\int_1^{\infty}\frac{[t]}{t^{s+1}}dt\\ &= \frac{s}{s-1} - s\int_1^{\infty}\frac{\{t\}}{t^{s+1}}dt\end{aligned}$$

where $[t]$ denotes the integer part of the real number $t$ and $\{t\}$ denotes its fractional part.

Since $\{t\} \in [0,1]$, the last integral converges for $\Re(s) > 0$. The right-hand side thus defines a meromorphic continuation of $\zeta(s)$ to the half-plane $\Re(s) > 0$.

The simple pole at $s = 1$ is clearly visible in equation (1.1). ☐

In order to extend the domain of the Riemann zeta function to the entire complex plane by the method of analytic continuation, we need to adopt a more systematic approach as in [**D**, 15.5]. This is accomplished step by step by extending the function first to the domain $\mathbf{D}_0 = \{z \in \mathbb{C} : \Re(z) > 0\}$, next to the domain $\mathbf{D}_{-1} = \{z \in \mathbb{C} : \Re(z) > -1\}$, and so on, adding a vertical strip to the domain at each step.

Observe that for each $n$, the function $g_n(z) = \frac{1}{n^{z-1}} - \frac{1}{(n+1)^{z-1}}$ is entire, with a zero at 1. Hence, for each $n$, $h_n$ defined by

$$h_n(z) = \frac{1}{z-1}\left(\frac{1}{n^{z-1}} - \frac{1}{(n+1)^{z-1}}\right)$$

is also entire. But for each $n$,

$$h_n(z) = \int_0^1 \frac{dt}{(n+t)^z} = \int_n^{n+1}\frac{dt}{t^z}. \tag{1.2}$$

Similarly, the function $f_n(z)$ defined by

$$\begin{aligned} f_n(z) &= \frac{1}{(n+1)^z} - h_n(z) \\ &= \frac{1}{(n+1)^z} - \int_0^1 \frac{dt}{(n+1)^z} \\ &= -z\int_0^1 \frac{tdt}{(n+1)^{z+1}} \end{aligned}$$

is also entire being the difference of two entire functions, as may be verified directly by integrating the last term by parts.

Now, for each $z$,

$$\begin{aligned} |f_n(z)| &\le |z| \cdot \int_0^1 \frac{tdt}{|(n+t)|^{z+1}} \\ &\le |z| \cdot \int_0^1 \frac{tdt}{(n+t)^{\Re(z)+1}} \end{aligned}$$

and if $z$ is such that $\Re(z) > \delta$ and $|z| < M$ for some $\delta > 0$ and $M > 0$, we have

$$|f_n(z)| \le M \cdot \frac{1}{n^{1+\delta}}$$

so that $\sum f_n$ converges uniformly on compact sets of $\mathbf{D}_0$ and hence defines a holomorphic function on $\mathbf{D}_0$. Note that while each $f_n$ is entire, $\sum f_n$ is holomorphic on $\mathbf{D}_0$ only.

On the other hand, we have for $z \in \mathbf{D}_1$,

$$\begin{aligned} \zeta(z) &= 1 + \frac{1}{2^z} + \frac{1}{3^z} + \cdots \\ &= 1 + \sum_{n=1}^{\infty} \frac{1}{(n+1)^z} \\ &= 1 + \sum_{n=1}^{\infty} \left( \int_0^1 \frac{dt}{(n+t)^z} + \frac{1}{(n+1)^z} - \int_0^1 \frac{dt}{(n+t)^z} \right) \\ &= 1 + \sum_{n=1}^{\infty} \int_0^1 \frac{dt}{(n+t)^z} - z \sum_{n=1}^{\infty} \int_0^1 \frac{tdt}{(n+t)^{z+1}}. \end{aligned}$$

Using equation (1.2), we have

$$\zeta(z) = 1 + \int_0^{\infty} \frac{dt}{t^z} - z \sum_{n=1}^{\infty} \int_0^1 \frac{tdt}{(n+t)^{z+1}}.$$

Thus,

$$\zeta(z) = 1 + \frac{1}{z-1} - z \sum_{n=1}^{\infty} \int_0^1 \frac{tdt}{(n+t)^{z+1}}. \tag{1.3}$$

Thus, for $z \in \mathbf{D}_1$, $\zeta(z)$ is given by the above expression. But in this expression the last term is a holomorphic function on $\mathbf{D}_0$ while the function $1 + 1/(z-1)$ is meromorphic on $\mathbf{D}_0$ with a simple pole at 1. Hence the function defined by the right-hand side of (1.3) is a meromorphic function on $\mathbf{D}_0$ with a simple pole at 1, and it coincides with the definition of the Riemann zeta function on $\mathbf{D}_1$. Hence we are justified if we define the zeta function on $\mathbf{D}_0$ by (1.3) above. This extends the zeta function from $\mathbf{D}_1$ to $\mathbf{D}_0$.

We proceed to continue the function to the domain

$$\mathbf{D}_{-1} = \{z \in \mathbb{C} : \Re(z) > -1\}.$$

The explicit formula is

$$\zeta(z) = 1 + \frac{1}{z-1} - \frac{z}{2!}[\zeta(z+1) - 1] - \frac{z(z+1)}{2!} \cdot \sum_{n=1}^{\infty} \int_0^1 \frac{t^2 dt}{(n+1)^{z+2}}$$

which is now holomorphic everywhere on $\mathbf{D}_{-1}$ except at 1, where it has a simple pole.

The next explicit formula is

$$\begin{aligned}\zeta(z) = 1 + \frac{1}{z-1} - \frac{z}{2!}(\zeta(z+1) - 1) - \frac{z(z+1)}{3!}(\zeta(z+2) - 1) \\ - \frac{z(z+1)(z+2)}{3!} \sum_{n=1}^{\infty} \int_0^1 \frac{t^3 dt}{(n+t)^{z+3}}\end{aligned}$$

which is now holomorphic on $\mathbf{D}_{-2} = \{z \in \mathbb{C} : \Re(z) > -2\}$, except of course at $s = 1$.

The technique to go from one explicit formula to the next is little more than integration by parts.

In this fashion, we define the Riemann zeta function over the entire complex plane. It is a meromorphic function, with a single, simple pole at $z = 1$. The residue of the zeta function at $z = 1$ is 1. Note that, at each step in the proof, the exponent of $(n + t)$ in the denominator of the integrand increases by 1.

**Definition 1.11.** The number $\rho \in \mathbb{C}$ is a zeta zero if $\zeta(\rho) = 0$.

**The Riemann hypothesis.** *All the zeta zeros in the open half-plane* $\Re(s) > 0$ *lie on the line* $\Re(s) = 1/2$.

The Riemann hypothesis is at present unproven. It is one of the most famous problems in analytic number theory, which holds the key to understanding the distribution of the prime numbers. The Riemann hypothesis is one of the Millenium Problems; see `www.claymath.org/millenium-problems`.

## Chapter 2

# The Euler product

## 2.1. The zeta function and the Euler product

In this chapter, we meet the Euler product

$$\prod \left(1 - p^{-s}\right)^{-1}$$

with $\Re(s) > 1$. This is an infinite product taken over all the prime numbers $p = 2, 3, 5, 7, \ldots$. We shall see that the Euler product is equal to $\zeta(s)$. This depends ultimately on the uniqueness of factorization for the natural numbers.

The Euler product can be traced back to Euler's seminal book *Introduction to Analysis of the Infinite* [**E**, Chapter XV]. In that work, Euler takes $s$ to be a positive integer. The product symbol $\prod$ was not available to Euler, and he wrote

$$\frac{1}{(1 - \frac{1}{2^n})(1 - \frac{1}{3^n})(1 - \frac{1}{5^n})(1 - \frac{1}{7^n}) \cdots}.$$

The Euler product is not immediately useful; however the *logarithm* of the Euler product is crucial. The logarithm of the Euler infinite product is an infinite series, the sum taken over all the prime numbers $p$. This infinite series is exploited later in this chapter to prove the important result that

$$\Re(s) = 1 \implies \zeta(s) \neq 0.$$

The logarithm of the Euler product is further exploited, later in this book (see Section 4.7) in proving the Riemann explicit formula.

For an informal proof of the equality of the Euler product with the zeta function, we can proceed as follows:

$$\begin{aligned}\prod_p \left(1-p^{-s}\right)^{-1} &= \prod_p \left(\sum_{n=0}^{\infty} p^{-ns}\right)\\ &= \sum_{n=1}^{\infty} \frac{1}{n^s}\\ &= \zeta(s).\end{aligned}$$

However, in order to take care of convergence issues, we may proceed as follows. We can think of the following theorem as the first theorem in analytic number theory. Note that Euler already had this theorem (with a forgivable typo) when $s$ is a positive integer, see [**E**, §275].

**Theorem 2.1.** *The Riemann zeta function $\zeta(s)$ is equal to the Euler product; that is,*

$$\zeta(s) = \prod_p \left(1-p^{-s}\right)^{-1}$$

*with* $\Re(s) > 1$.

**Proof.** Recall that for $\Re(s) > 1$,

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots + \frac{1}{n^s} + \cdots .$$

Therefore we have

$$\left(1-\frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \cdots + \frac{1}{(2n+1)^s} + \cdots$$

as all the even terms cancel each other. Then

$$\begin{aligned}\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{2^s}\right)\zeta(s) &= \left(1-\frac{1}{3^s}\right)\left(1+\frac{1}{3^s}+\frac{1}{5^s}+\frac{1}{7^s}+\cdots\right)\\ &= 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \cdots .\end{aligned}$$

This time all terms having 3 as factor in the denominator cancel each other; this is an example of the *sieve of Eratosthenes.* Proceeding in this way, if $p_1, p_2, \ldots, p_k$ are the first $k$ primes,

$$\prod_{j=1}^{k}\left(1-\frac{1}{p_j^s}\right)\zeta(s)-1=\sum_{m}\frac{1}{m^s}$$

with summation over all natural numbers $m$ which do not have $p_1$, $p_2, \ldots, p_k$ as a factor.

Let $q_k$ be the least such number, and let $\Re(s)=1+\delta$ with $\delta>0$. Then we have

$$\left|\prod_{j=1}^{k}\left(1-\frac{1}{p_j^s}\right)\zeta(s)-1\right|\leq\sum_{m\geq q_k}\frac{1}{m^{1+\delta}}.$$

The term on the right-hand side is the tail of the infinite series for $\zeta(1+\delta)$. Now let $k\to\infty$. We get

$$\prod_{j=1}^{\infty}\left(1-\frac{1}{p_j^s}\right)\cdot\zeta(s)=1$$

and the convergence is locally uniform in $\Re(s)>1$. This gives Euler's theorem for $\Re(s)>1$

$$\zeta(s)=\prod_{p}\left(1-\frac{1}{p^s}\right)^{-1}$$

where the product on the right is over all the primes. ☐

This proof provides us with a bonus, in the following way.

**Theorem 2.2.** *The Riemann zeta function* $\zeta(s)\neq 0$ *in the open half-plane* $\Re(s)>1$.

**Proof.** Given $0<\varepsilon<1$, there exists $k$ such that

$$\left|\prod_{j=1}^{k}\left(1-\frac{1}{p_j^s}\right)\zeta(s)-1\right|<\varepsilon.$$

This inequality forces $\zeta$ to be nonzero in $\Re(s)>1$. ☐

## 2.2. The logarithmic derivative of $\zeta(s)$

We now consider the logarithmic derivative of the Riemann zeta function. First, we need to consider what the derivative of an infinite product is. Suppose that $f$ is a function that can be written as

$$f = \prod_{i=1}^{\infty} g_i.$$

So what can we say about the derivative $df/dx$? To get an idea how this works, let's first consider the derivative of finite products of functions. By the familiar Leibniz rule for products, we have

$$\frac{d\,(g_1 g_2)}{dx} = g_2 \frac{dg_1}{dx} + g_1 \frac{dg_2}{dx}$$

and

$$\frac{d\,(g_1 g_2 g_3)}{dx} = g_2 g_3 \frac{dg_1}{dx} + g_1 g_3 \frac{dg_2}{dx} + g_1 g_2 \frac{dg_3}{dx}.$$

So we have

$$\begin{aligned} f &= g_1 g_2 g_3 \\ \Longrightarrow \frac{1}{f}\left(\frac{df}{dx}\right) &= \frac{1}{g_1}\left(\frac{dg_1}{dx}\right) + \frac{1}{g_2}\left(\frac{dg_2}{dx}\right) + \frac{1}{g_3}\left(\frac{dg_3}{dx}\right). \end{aligned}$$

Similarly, for a finite product we get that

$$\begin{aligned} f &= \prod_{i=1}^{n} g_i \\ \Longrightarrow \frac{1}{f}\left(\frac{df}{dx}\right) &= \sum_{i=1}^{n} \frac{1}{g_i}\left(\frac{dg_i}{dx}\right). \end{aligned}$$

This is a good pointer as to what happens for infinite products. We now move on to infinite products.

**Theorem 2.3.** *Let $U$ be an open set in $\mathbb{C}$. Suppose that*

$$f = \prod_{i=1}^{\infty} g_i$$

*with $g_i \in \mathrm{Hol}(U)$, $f \neq 0$, $g_i \neq 0$ in $U$ for all $i \geq 1$, and suppose that the convergence is locally uniform in $U$. Then we have*

$$\frac{1}{f}\left(\frac{df}{dz}\right) = \sum_{i=1}^{\infty} \frac{1}{g_i}\left(\frac{dg_i}{dz}\right).$$

**Proof.** Let $f_n = \prod_{i=1}^{n} g_i$ be the product of the the first $n$ functions. We have that

$$\frac{1}{f_n}\left(\frac{df_n}{dz}\right) = \sum_{i=1}^{n} \frac{1}{g_i}\left(\frac{dg_i}{dz}\right).$$

By the convergence lemma, we know that $f \in \mathrm{Hol}(U)$ and that $df_n/dz \longrightarrow df/dz$ as $n \longrightarrow \infty$. Let $C \subseteq U$ be a circle. Since $C$ is a compact subset of $U$ and $|f(z)| > 0$ on $C$, there exists $\delta > 0$ such that $|f(z)| \geq \delta > 0$ for all $z \in C$. Let $N$ be such that $|f_n(z)| \geq \delta/2$ for all $z \in C$ and for $n \geq N$. Then, for $z \in C$ and $n \geq N$,

$$\begin{aligned}\left|\frac{1}{f_n(z)} - \frac{1}{f(z)}\right| &= \frac{|f_n(z) - f(z)|}{|f_n(z) \cdot f(z)|} \\ &\leq \frac{2}{\delta^2}\left|f_n(z) - f(z)\right|.\end{aligned}$$

So $1/f_n \longrightarrow 1/f$ uniformly on the circle $C$. Hence, by the maximum modulus theorem [**Des**, p. 144] we have that $1/f_n \longrightarrow 1/f$ uniformly on every compact disc $D \subseteq U$.

At this stage of the argument, we know that $1/f_n$ converges locally uniformly to $1/f$ and $df_n/dz$ converges locally uniformly to $df/dz$. Combining these two results, we infer that the sequence $f_n'/f_n$ converges locally uniformly to $f'/f$.

Hence

$$\frac{1}{f}\left(\frac{df}{dz}\right) = \sum_{i=1}^{\infty} \frac{1}{g_i}\left(\frac{dg_i}{dz}\right)$$

as required. ☐

If we have a holomorphic function $F(z)$ in a domain $U$, then we can differentiate *after* taking the logarithm:

$$\frac{d}{dz}(\log F(z)) = \frac{F'(z)}{F(z)}.$$

This expression is called the logarithmic derivative of $F$.

The reward for our recent rather technical excursion is the following result on the logarithmic derivative of the zeta function.

**Lemma 2.4.** *For the logarithmic derivative of the zeta function, we have, for* $\Re(s) > 1$,

$$\begin{aligned}
\frac{1}{\zeta(s)}\left(\frac{d\zeta}{ds}\right) &= \frac{1}{\zeta(s)}\left(\frac{d}{ds}\left(\prod_p \left(1-p^{-s}\right)^{-1}\right)\right) \\
&= \sum_p \frac{d\left(1-p^{-s}\right)^{-1}/ds}{\left(1-p^{-s}\right)^{-1}} \\
&= -\sum_p \frac{\left(1-p^{-s}\right)^{-2} p^{-s}\log p}{\left(1-p^{-s}\right)^{-1}} \\
&= -\sum_p \left(1-p^{-s}\right)^{-1} p^{-s}\log p \\
&= -\sum_p \frac{p^{-s}\log p}{\left(1-p^{-s}\right)} \\
&= -\sum_p \frac{\log p}{p^s-1}.
\end{aligned}$$

We now introduce the von Mangoldt function.

**Definition 2.5.** We define the von Mangoldt function $\Lambda(n)$ as follows:

$$\begin{aligned}
\Lambda(n) &= \log p \quad \text{if} \quad n \quad \text{is a power of} \quad p \\
&= 0 \quad \text{otherwise.}
\end{aligned}$$

**Theorem 2.6.** *The logarithmic derivative of the Riemann zeta function in the open half-plane* $\Re(s) > 1$ *is given by*

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

**Proof.** Suppose that $s \in \mathbb{C}$ and consider the zeta function $\zeta(s)$ in the open half-plane $\Re(s) > 1$. Now by Lemma 2.4, we have that

$$\begin{aligned}\frac{\zeta'(s)}{\zeta(s)} &= \sum_p -\log p\left(\frac{p^{-s}}{1-p^{-s}}\right)\\ &= \sum_p -\log p\sum_{k=1}^{\infty} p^{-ks}\\ &= -\log 2\sum_{k=1}^{\infty} 2^{-ks} - \log 3\sum_{k=1}^{\infty} 3^{-ks} - \log 5\sum_{k=1}^{\infty} 5^{-ks} - \cdots\\ &= -\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}.\end{aligned}$$

This completes the proof of the theorem. □

We know that the series $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent. Suppose that we delete all composite numbers $n$, leaving the series $\sum_p \frac{1}{p}$ where $p$ runs through all the prime numbers. Is this series still divergent?

**Theorem 2.7.** *The series* $\sum_p \frac{1}{p}$ *is divergent; that is,*

$$\sum_p \frac{1}{p} = \infty.$$

**Proof.** Inspired by [**TMF**, p. 13], write $n = qm^2$ where $q$ denotes a square-free integer. We have

$$\begin{aligned}\sum_{n\le x}\frac{1}{n} &\le \sum_{q\le x}\frac{1}{q}\sum_{m\le\sqrt{x/q}}\frac{1}{m^2}\\ &< \frac{\pi^2}{6}\sum_{q\le x}\frac{1}{q}\\ &< 2\sum_{q\le x}\frac{1}{q}.\end{aligned}$$

We also have

$$\begin{aligned}\sum_{q\le x}\frac{1}{q} &\le \prod_{p\le x}(1+1/p)\\ &\le \exp\left(\sum_{p\le x}\frac{1}{p}\right)\end{aligned}$$

and

$$\begin{aligned}\sum_{n\le x}\frac{1}{n} &\ge \sum_{n\le x}\int_n^{n+1}\frac{dt}{t}\\ &\ge \log x.\end{aligned}$$

We neatly combine all three inequalities:

$$\sum_{p\le x}\frac{1}{p} > \log\log x - \log 2$$

for all $x \ge 2$. This implies that $\sum_p 1/p = \infty$. □

So we do indeed have that the series $\sum_p 1/p = \infty$ is divergent, but the divergence is very slow. Euler would probably have written

$$\sum \frac{1}{p} = \log\log\infty.$$

**Theorem 2.8.** *Suppose that $s = \sigma + it \in \mathbb{C}$. The Riemann zeta function $\zeta(s) \ne 0$ on the line $\sigma = 1$.*

**Proof.** Suppose that $s = \sigma + it$ and consider the zeta function $\zeta(s)$ in the open half-plane $\sigma > 1$. So we have that

$$\begin{aligned}\log\zeta(s) &= \sum_p \log\left(1-p^{-s}\right)^{-1}\\ &= \sum_p\sum_{m=1}^{\infty}\frac{1}{m}p^{-ms}\\ &= \sum_p\sum_{m=1}^{\infty}\frac{1}{m}p^{-m\sigma}e^{itm\log p}\end{aligned}$$

and so $\Re(\log\zeta(s))$ is given by

$$\sum_p\sum_{m=1}^{\infty}\frac{1}{m}p^{-m\sigma}\cos(tm\log p) = \sum_p\sum_{m=1}^{\infty}\frac{1}{m}p^{-m\sigma}\cos\left(t\log p^m\right).$$

We now use a simple trigonometric identity:

$$\begin{aligned}3+4\cos\varphi+\cos 2\varphi &= 3+4\cos\varphi+2\cos^2\varphi-1\\ &= 2+4\cos\varphi+2\cos^2\varphi\\ &= 2(1+\cos\varphi)^2\\ &\geq 0.\end{aligned}$$

Set $\varphi = t\log p^m$ and we immediately have

$$3+4\cos\left(t\log p^m\right)+\cos\left(2t\log p^m\right)\geq 0.$$

Now we sum all these nonnegative terms:

$$\sum_p\sum_{m=1}^{\infty}\frac{p^{-m\sigma}}{m}\left(3+4\cos(t\log p^m)+\cos(2t\log p^m)\right)\geq 0$$

which is equal to

$$3\cdot\log\zeta(\sigma)+4\Re\left(\log\zeta\left(\sigma+it\right)\right)+\Re\left(\log\zeta\left(\sigma+2it\right)\right)\geq 0.$$

This statement is equivalent to the following inequality:

$$\zeta^3\left(\sigma\right)\left|\zeta^4\left(\sigma+it\right)\right|\left|\zeta\left(\sigma+2it\right)\right|\geq 1.$$

This depends on the fact that

$$\Re(\log z)=\log|z|$$

which follows from the definition of $\log z$:

$$\log z=\log|z|+i\arg z.$$

At this point, we can be sure that

$$(\sigma-1)^3\zeta^3(\sigma)\left|\frac{\zeta\left(\sigma+it\right)}{\sigma-1}\right|^4\left|\zeta\left(\sigma+2it\right)\right|\geq\frac{1}{\sigma-1}. \tag{2.1}$$

Now suppose that $\zeta\left(1+it\right)=0$, with $t\neq 0$. Recall that $\zeta(s)$ has a simple pole at $s=1$ and so $(\sigma-1)\zeta(\sigma)\to 1$ as $\sigma\longrightarrow 1$ from the right.

Now let $\sigma \longrightarrow 1$ from the right. We have

$$\begin{aligned} (\sigma-1)^3\zeta^3(\sigma) &\longrightarrow 1, \\ |\zeta\,(\sigma+2it)| &\longrightarrow |\zeta(1+2it)|, \\ \left|\frac{\zeta\,(\sigma+it)}{\sigma-1}\right| = \left|\frac{\zeta\,(\sigma+it)-\zeta\,(1+it)}{\sigma-1}\right| &\longrightarrow \zeta'(1+it), \end{aligned}$$

and so, as $\sigma \longrightarrow 1$ from the right, the left-hand side of (2.1) tends to a finite limit. This contradicts the fact that the right-hand side of (2.1) tends to $\infty$. Hence $\zeta\,(s) \neq 0$ on the line $\sigma = 1$ as required. $\square$

We now make an elementary but slightly subtle observation. The critical strip is defined as $\{s \in \mathbb{C} : 0 \leq \Re(s) \leq 1\}$. Suppose that we take a rectangle **D**, of finite area, in the critical strip, with its right-hand edge on the line $\Re(s) = 1$. There will only be a finite number of zeta zeros within this rectangle. By narrowing the width of this rectangle to avoid all of these zeros, we can construct a new rectangle **E**, also of finite area, in the critical strip, its right-hand edge on $\Re(s) = 1$, which is completely free of zeta zeros.

In this precise sense, Theorem 2.8 allows us to extend the zero-free region of the zeta function into the critical strip.

## Chapter 3

# The functional equation

We revisit the zeta function in the open half-plane $\Re(s) > 1$. In this domain, the zeta function has an evident reflection symmetry:

$$\begin{aligned}\overline{\zeta(s)} &= \overline{\sum_{n=1}^{\infty} \frac{1}{n^s}} \\ &= \sum_{n=1}^{\infty} \frac{1}{n^{\overline{s}}} \\ &= \zeta(\overline{s})\end{aligned}$$

where the overline denotes complex conjugation.

The meromorphic continuation of the Riemann zeta function $\zeta(s)$ into the open half-plane $\Re(s) > 0$ is given by

$$\zeta(s) = 1 + \frac{1}{s-1} - s\int_1^{\infty} \frac{\{x\}}{x^{s+1}} dx;$$

see Theorem 1.10. Taking complex conjugates, we obtain

$$\begin{aligned}\overline{\zeta(s)} &= 1 + \frac{1}{\overline{s}-1} - \overline{s}\int_1^{\infty} \frac{\{x\}}{x^{\overline{s}+1}} dx \\ &= \zeta(\overline{s}).\end{aligned}$$

The *critical strip* is defined to be the open set

$$\{s : 0 \leq \Re(s) \leq 1\}$$

and the *critical line* is defined to be the line

$$\{s : \Re(s) = 1/2\}.$$

So this reflection symmetry continues into the critical strip. We have

$$\zeta(s) = 0 \implies \zeta(\overline{s}) = 0.$$

In the critical strip, the zeta zeros occur in complex conjugate pairs. In particular, if $\rho = 1/2 + i\gamma$ is a zeta zero on the critical line, then $\overline{\rho} = 1/2 - i\gamma$ is also a zeta zero on the critical line.

The zeta function has, in addition to this symmetry, a hidden symmetry, which only becomes evident after an overhaul of the zeta function. To carry out this overhaul, we have to multiply $\zeta(s)$ by

$$\frac{1}{2}s(s-1)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}$$

where $\Gamma(s)$ is the gamma function. The resulting function will be denoted $\xi(s)$. This new function has the following symmetry:

$$\xi(s) = \xi(1-s). \tag{3.1}$$

The map $s \mapsto 1 - s$ is reflection in the horizontal line $\Im(s) = 0$ followed by reflection in the critical line $\Re(s) = 1/2$.

The equation 3.1) is the functional equation of Riemann, and appears in his memoir [**R**] of 1859. The functional equation was anticipated by Euler. Euler's paper, written in 1749, appeared in 1768 in the *Memoirs* of the Berlin Academy; it was never reprinted. For a hundred years after their discovery, Euler's functional equations were utterly forgotten; see [**We**, p. 276].

In this chapter, we will do many things. We will define the gamma function $\Gamma(s)$ and establish its basic properties. We will then establish the Euler-Riemann functional equation, using it to locate the trivial zeros and to compute a few zeta values. We will then give an account of Euler's great work of 1749.

We will encounter the Euler constant, which reappears in several places:

- in the definition of the gamma function;
- as the constant term in the Laurent expansion of $\zeta(s)$ around $s = 1$;
- in the linear term of the Taylor expansion of $\Gamma(s)$ around $s = 1$.

We make an excursion into the realm of the Kontsevich-Zagier periods, with the aim of providing a context for the Euler constant.

## 3.1. The gamma function

**Definition 3.1.** If $n$ is a natural number, then $n!$ is defined as

$$n! = 1 \cdot 2 \cdot 3 \cdots n.$$

To this we add the convention that $0! = 1$.

Recall from Chapter 1 that the Riemann zeta function started life as $\zeta(k)$ where $k \in \mathbb{N}$. In other words, it was originally defined by Euler as

$$\zeta\left(k\right) = \sum_{n=1}^{\infty} \frac{1}{n^k}.$$

We then extended the definition to $\zeta(s)$ where $s \in \mathbb{C}$. To define the gamma function $\Gamma(s)$, what we actually do is extend the definition of the factorial function $n!$ over the complex numbers $s \in \mathbb{C}$.

First of all, we define

$$I_n = \int_0^\infty e^{-t} t^n dt$$

with $n \in \mathbb{N}$. Integrating by parts, we have

$$\begin{aligned} I_n &= \left[-e^{-t} t^n\right]_0^\infty + \int_0^\infty n e^{-t} t^{n-1} dt \\ &= I_{n-1}. \end{aligned}$$

Since

$$I_0 = \int_0^\infty e^{-t} dt = 1,$$

we can agree that

$$I_n = n!.$$

This equation is a stepping stone towards a definition of the gamma function.

**Definition 3.2.** Given $z \in \mathbb{C}$, the gamma function $\Gamma(z)$ is defined by the integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

Note that $\Gamma(n+1) = n!$

The integral

$$\int_0^\infty e^{-t} t^{z-1} dt$$

is convergent if $\Re(z) > 1$ and defines a holomorphic function in that domain. Let

$$\mathbf{D}_n = \{z : \Re(z) > n\}.$$

With $z \in \mathbf{D}_1$ we have $\Gamma(z+1) = z\Gamma(z)$ and so

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}.$$

This leads us to define the gamma function in $\mathbf{D}_0$ as follows:

$$\Gamma(z) := \frac{\Gamma(z+1)}{z}, \qquad z \in \mathbf{D}_0.$$

Iterating this, we define

$$\Gamma(z) := \frac{\Gamma(z+1)}{z}, \qquad z \in \mathbf{D}_{-1},$$

and so on through all the domains $\mathbf{D}_{-1} \subset \mathbf{D}_{-2} \subset \mathbf{D}_{-3} \subset \ldots$.

In this iterative procedure, do we encounter any singularities? We have

$$\begin{aligned}\Gamma(z) &= \int_0^\infty e^{-t} t^{z-1} dt \\ &= \int_0^1 e^{-t} t^{z-1} dt + \int_1^\infty e^{-t} t^{z-1} dt \\ &= \Gamma_1(z) + \Gamma_2(z)\end{aligned}$$

where

$$\Gamma_1(z) = \int_0^1 e^{-t} t^{z-1} dt$$

and

$$\Gamma_2(z) = \int_1^\infty e^{-t} t^{z-1} dt.$$

Now $\Gamma_2(z)$ is an entire function; in other words it is holomorphic in the whole of the complex plane. So the singularities of the gamma function are the singularities of $\Gamma_1(z)$:

$$\begin{aligned}\Gamma_1(z) &= \int_0^1 e^{-t} t^{z-1} dt \\ &= \int_0^1 t^{z-1} \left( \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} \right) dt \\ &= \sum_{n=0}^{\infty} \int_0^1 \frac{(-1)^n t^{z+(n-1)}}{n!} dt \\ &= \sum_{n=0}^{\infty} \left[ \frac{(-1)^n t^{z+n}}{n!(z+n)} \right]_0^1 \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(z+n)}.\end{aligned}$$

So we have simple poles at

$$z = 0, -1, -2, -3, \ldots.$$

**Theorem 3.3.** *Each term in the above expression reveals a simple pole of the gamma function* $\Gamma(z)$ *at the points* $0, -1, -2, ..., -n, ...$ *and the residue at* $z = -n$ *is* $(-1)^n / n!$.

Restricting to a real variable $x$, we still have

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}$$

and it is instructive to draw the graph of $y = \Gamma(x)$ for all $x \in \mathbb{R}$.

For example, let $\varepsilon$ be small and positive. Then we have

$$\Gamma(\varepsilon) = \frac{\Gamma(1+\epsilon)}{\varepsilon} \approx \frac{1}{\varepsilon}$$

which is large and positive. At the same time,

$$\Gamma(-\varepsilon) = \frac{\Gamma(1-\varepsilon)}{-\varepsilon} \approx \frac{1}{-\varepsilon}$$

which is large and negative. The gamma function $y = \Gamma(x)$ has a severe discontinuity at $x = 0$, namely a jump discontuity with infinite jump.

In general, and for the same reason, the gamma function $y = \Gamma(x)$ has jump discontinuities with infinite jump at each negative integer $0, -1, -2, -3, \ldots$.

In addition, the vertical lines $\{s : \Re(s) = n\}$ with $n = 0, -1, -2, -3, \ldots$ are asymptotes of the curve $y = \Gamma(x)$.

*The simple poles of* $\Gamma(z)$*, which occur at* $z = 0, -1, -2, -3, \ldots$*, correspond to the jump discontinuities of the curve* $y = \Gamma(x)$*, which occur at* $x = 0, -1, -2, -3, \ldots$.

The point of view of Emil Artin is "that the gamma function $\Gamma(x)$ can be thought of as one of the elementary functions, and that all of its basic properties can be established using elementary methods of the calculus." See his monograph [**Art**].

**3.1.1. An alternative approach.** This approach is due to Deshpande [**Des**, p. 225]. Consider the function $\phi$ defined by the infinite product

$$\phi(z) = \prod_{j=1}^{\infty} (1 + z/j) \exp(-z/j).$$

This infinite product satisfies the hypothesis of the Weierstrass factorization theorem [**Des**, Prop. 15.14] and so defines an entire function wtih simple zeros at $-1, -2, -3, \ldots$. The function $\psi(z) = z\phi(z)$ is also entire with simple zeros at $0, -1, -2, \ldots$. Let $\phi_k$ denote the finite product

$$\phi_k(z) = \prod_{j=1}^{k} (1 + z/j) \exp(-z/j).$$

It is an entire function for each $k = 1, 2, \ldots$ and $\lim \phi_k = \phi$. Routine computation shows that

$$\begin{aligned}\frac{\phi_k'(z)}{\phi_k(z)} &= \sum_{n=1}^{k} \frac{(1/n) - (1/n)(1 + z/n)}{1 + (z/n)} \\ &= \sum_{n=1}^{k} \left(\frac{1}{n+z} - \frac{1}{n}\right).\end{aligned}$$

Similarly,

$$\frac{\phi_k'(z-1)}{\phi_k(z-1)} = \sum_{n=1}^{k} \left(\frac{1}{n-1+z} - \frac{1}{n}\right)$$

so that

$$\frac{\phi_k'(z-1)}{\phi_k(z-1)} - \frac{\phi_k'(z)}{\phi_k(z)} = \frac{1}{z} - \frac{1}{k+z}. \tag{3.2}$$

We use this identity in the next result.

**Lemma 3.4.** *Let $\phi$ be the function defined above. Then there is a constant $\gamma$ such that for any $z$,*

$$\phi(z-1) = \exp(\gamma) \cdot z\phi(z).$$

**Proof.** The function $\psi_1(z) = z\phi(z)$ is entire with simple zeros at $0, -1, -2, -3, \ldots$. So is the function $\psi_2(z) = \phi(z-1)$ with the same simple zeros. Hence by the Weierstrass factorization theorem there is an entire function $\gamma$ such that $\phi(z-1) = \exp(\gamma(z)) \cdot z\phi(z)$. We now show that $\gamma$ is a constant function. For this, it suffices to show that $\gamma' = 0$.

Differentiating both sides of $\phi(z-1) = \exp(\gamma(z)) \cdot z\phi(z)$, we get

$$\begin{aligned}&\phi'(z-1) = \exp(\gamma(z))(\phi(z) + z\phi'(z) + \gamma'(z)z\phi(z)), \\ &\frac{\phi'(z-1)}{\phi(z-1)} = \frac{1}{z} + \frac{\phi'(z)}{\phi(z)} + \gamma'(z).\end{aligned}$$

From this we get

$$\gamma'(z) = \frac{\phi'(z-1)}{\phi(z-1)} - \frac{\phi'(z)}{\phi(z)} - \frac{1}{z}. \tag{3.3}$$

On the other hand, we have by (3.2)

$$\frac{\phi_k'(z-1)}{\phi_k(z-1)} - \frac{\phi_k'(z)}{\phi_k(z)} = \frac{1}{z} - \frac{1}{k+z}.$$

Now differentiation is a continuous operator on entire functions. Hence $\lim \phi_k' = \phi'$ since $\lim \phi_k = \phi$. Hence

$$\begin{aligned}\lim_{k\to\infty}\left(\frac{\phi_k'(z-1)}{\phi_k(z-1)} - \frac{\phi_k'(z)}{\phi_k(z)}\right) &= \lim_{k\to\infty}\left(\frac{1}{z} - \frac{1}{k+z}\right)\\ &= \frac{1}{z}\\ &= \frac{\phi'(z-1)}{\phi(z-1)} - \frac{\phi'(z)}{\phi(z)}.\end{aligned}$$

From (3.3) we obtain immediately that

$$\gamma'(z) = 0$$

so that $\gamma$ is a constant function. ☐

The constant $\gamma$ so obtained is known as the Euler constant. This is its first appearance in this book. To evaluate it, note that $\phi(0) = 1$ by the definition of $\phi$.

We have

$$\begin{aligned}\phi(1) &= \prod_{j=1}^{\infty}(1 + 1/j)\exp(-1/j)\\ &= \lim c_n\end{aligned}$$

where

$$\begin{aligned}c_n &= \prod_{j=1}^{n}(1 + 1/j)\exp(-1/j)\\ &= \exp\left(-\sum_{j=1}^{n}(1/j)\right)\prod_{j=1}^{n}(j+1)/j.\end{aligned}$$

This gives

$$\begin{aligned}
\log c_n &= -\sum_{j=1}^{n}(1/j) + \sum_{j=1}^{n}(\log(j+1) - \log j),\\
-\log c_n &= \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) - \log n - \log((n+1)/n),\\
-\lim_{n\to\infty} \log c_n &= \lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right).
\end{aligned}$$

Letting $z = 1$ in the equation $\phi(z-1) = \exp(\gamma)\cdot z\phi(z)$, we have

$$\begin{aligned}
\phi(0) &= \exp(\gamma)\cdot\phi(1),\\
\log\phi(0) &= \gamma + \log\phi(1),\\
0 &= \gamma + \log(\lim c_n),\\
\gamma &= -\lim\log c_n,
\end{aligned}$$

so that

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\right).$$

Recall that $\phi$ is defined by

$$\phi(z) = \prod_{j=1}^{\infty}(1 + z/j)\exp(-z/j).$$

It has simple zeros at $-1, -2, -3, \ldots$ and it satisfies the relation

$$\phi(z-1) = e^{\gamma} z\phi(z).$$

The gamma function is now defined by

$$\Gamma(z) = \frac{\exp(-\gamma z)}{z\phi(z)}$$

for $z \neq 0, -1, -2, \ldots$ where $\gamma$ is the Euler constant. It is clearly a meromorphic function with simple poles at $0, -1, -2, \ldots$. It has no zeros.

From the recurrence relation of $\phi$, we have

$$(z+1)\phi(z+1) = e^{-\gamma}\phi(z).$$

Substituting this in

$$\Gamma(z+1) = \frac{e^{-\gamma(z+1)}}{(z+1)\phi(z+1)}$$

gives

$$\begin{aligned}\Gamma(z+1) &= \frac{e^{-\gamma(z+1)}}{e^{-\gamma}\phi(z)} \\ &= \frac{e^{-\gamma z}}{\phi(z)} \\ &= z\Gamma(z)\end{aligned}$$

which is the recurrence relation for $\Gamma$.

*The gamma function* $\Gamma(z)$ *is a meromorphic function in the complex plane* $\mathbb{C}$ *with simple poles at* $0, -1, -2, -3, \ldots$. *The gamma function is zero-free; i.e., it never takes the value* $0$. *We have* $\Gamma(n+1) = n!$.

## 3.2. The functional equation

Recall that the zeta function started life, in the work of Euler, as a function of natural numbers; that is, $\zeta\left(k\right) = \sum_{n=1}^{\infty} \frac{1}{n^k}$. This definition was then extended to the absolutely convergent series

$$\zeta\left(s\right) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad\qquad \Re(s) > 1,$$

which was then extended to a meromorphic function in the whole of the complex plane, with a simple pole at $s = 1$.

Then at the beginning of this section we saw that the gamma function started life as the factorial function $n!$. This can then be extended as a meromorphic function into the whole of the complex plane, with simple poles at the negative integers $0, -1, -2, -3, \ldots$. These two functions are linked by a single equation called the functional equation, and from this we can uncover the symmetries of the zeta zeros.

The following theorem is one of the most important theorems in mathematics.

**Theorem 3.5** (Analytic continuation of the completed zeta function)**.** *Define the completed zeta function by*

$$\xi(s) = \frac{1}{2}s(s-1)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s).$$

*Then $\xi(s)$, originally defined for $\Re(s) > 1$, has an analytic continuation to an entire function and satisfies the functional equation*

$$\xi(s) = \xi(1-s).$$

So in full we have

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta\left(s\right) = \pi^{(s-1)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta\left(1-s\right). \tag{3.4}$$

**Proof.** For the functional equation we follow Riemann's original argument as described in [**MTB**, 3.1.19]. For $\Re(s) > 0$, by definition of the gamma function as a definite integral, and the change of variables $t = n^2\pi x$, we have

$$\int_0^\infty x^{\frac{s}{2}-1}e^{-n^2\pi x}dx = \frac{\Gamma(\frac{s}{2})}{n^s\pi^{\frac{s}{2}}}.$$

Summing over $n \in \mathbb{N}$, with $\Re(s) > 1$ to guarantee convergence, we obtain

$$\begin{aligned}
\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \int_0^\infty x^{\frac{s}{2}-1}\left(\sum_{n=1}^\infty e^{-n^2\pi x}\right)dx \\
&= \int_0^\infty x^{\frac{s}{2}-1}\omega(x)dx \\
&= \int_1^\infty x^{\frac{s}{2}-1}\omega(x)dx + \int_0^1 x^{\frac{s}{2}-1}\omega(x)dx \\
&= \int_1^\infty x^{\frac{s}{2}-1}\omega(x)dx + \int_1^\infty x^{-\frac{s}{2}-1}\omega\left(\frac{1}{x}\right)dx
\end{aligned}$$

where $\omega(x) = \sum_{n=1}^\infty e^{-n^2\pi x}$. Note that we divided the integral into two pieces and changed variables by $x \mapsto x^{-1}$ in the second integral; this leads to rapidly converging integrals. The absolute convergence of the sum-integral justifies the rearrangement of terms. We will show in a moment that the function $\omega$ satisfies the functional equation

$$\omega\left(\frac{1}{x}\right) = -\frac{1}{2} - \frac{1}{2}x^{1/2} + x^{1/2}\omega(x)$$

for $x > 0$. Simple algebra then shows that for $\Re(s) > 1$ we have

$$\pi^{-\frac{1}{2}s}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty (x^{\frac{1}{2}s-1} + x^{-\frac{1}{2}s-\frac{1}{2}})\omega(x)dx. \tag{3.5}$$

Because of the rapid decay of $\omega(x)$, the integral on the right converges absolutely for *any* $s$ and represents an entire function of $s$. The remaining assertions of the theorem are easy consequences of the location of poles of $1/s(s-1)$ and the invariance of the right-hand side of (3.5) under $s \mapsto 1-s$. It remains to verify the functional equation of $\omega$. For this we write

$$\omega(x) = \frac{\theta(x)-1}{2}$$

with

$$\theta(x) = \sum_{n=-\infty}^{+\infty} e^{-\pi n^2 x}.$$

Now this series is rapidly converging for $x > 0$. The desired functional equation for $\omega$ easily follows from

$$\theta(x^{-1}) = x^{\frac{1}{2}}\theta(x), \qquad x > 0,$$

which we now prove.

Hold $x$ constant and introduce the function $f$ as follows:

$$f(t) = e^{-\pi t^2 x^{-1}}.$$

This is a standard Gaussian curve of the form $t \mapsto e^{-at^2}$. The Fourier transform of $e^{-at^2}$ is $y \to \sqrt{\pi/a} \cdot e^{-\pi^2 y^2/a}$. With $a = \pi/x$ the Fourier transform of $f$ is given by

$$\widehat{f}(y) = \sqrt{x} \cdot e^{-\pi y^2 x}$$

and, in particular, we have

$$\widehat{f}(n) = \sqrt{x} \cdot e^{-\pi n^2 x}.$$

Now apply the Poisson summation formula [**MTB**, §11.4.2]. We get

$$\begin{aligned}\theta(x^{-1}) &= \sum_{-\infty}^{+\infty} f(n)\\ &= \sum_{-\infty}^{+\infty} \widehat{f}(n)\\ &= x^{\frac{1}{2}}\theta(x)\end{aligned}$$

as required. ☐

The functional equation can also be expressed asymmetrically [**MV**, p. 329].

**Corollary 3.6.** *For all* $s \neq 1$,

$$\zeta(s) = \zeta(1-s)2^s\pi^{s-1}\Gamma(1-s)\sin(\pi s/2).$$

**Proof.** We need the reflection principle (C.6) and the duplication formula (C.9) in [**MV**]:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$$

and

$$\Gamma(s)\Gamma(s+1/2) = \sqrt{\pi}2^{1-2s}\Gamma(2s).$$

Applying these two formulas to the symmetrical form of the functional equation (3.4), we obtain

$$\begin{aligned}\zeta(s) &= \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\zeta(1-s)\pi^{s-1/2}\\ &= \Gamma\left(\frac{1-s}{2}\right)\cdot\Gamma\left(1-\frac{s}{2}\right)\frac{\sin(\pi s/2)}{\pi}\cdot\zeta(1-s)\pi^{s-1/2}\\ &= \sqrt{\pi}2^s\Gamma(1-s)\cdot\frac{\sin(\pi s/2)}{\pi}\cdot\zeta(1-s)\pi^{s-1/2}\\ &= \zeta(1-s)2^s\pi^{s-1}\Gamma(1-s)\sin(\pi s/2).\end{aligned}$$

☐

We now look for zeros of the Riemann zeta function $\zeta(s)$ in the open half-plane $\Re(s) < 0$. If $\Re(s) < 0$, then $\Re(1-s) > 1$. In this domain, the right-hand side of the functional equation is holomorphic and zero-free. Therefore, in the domain $\Re(s) < 0$, the left-hand side of the functional equation is holomorphic and zero-free.

The simple poles of $\Gamma\left(s/2\right)$ at the points $s = -2, -4, -6, -8, ...$ must therefore be canceled by simple zeros of the zeta function at these points. These are the *trivial zeros* of the zeta function.

**Theorem 3.7.** *The Riemann zeta function $\zeta\left(s\right)$ has simple zeros at the negative even integers given by*

$$s = -2, -4, -6, \ldots .$$

*These are the trivial zeros of the zeta function.*

So we know that all the zeros of the Riemann zeta function are trivial in the open half-plane $\Re(s) < 0$, and we have proved in Chapter 2 that the zeta function was zero-free in the closed half-plane $\Re(s) \geq 1$.

What about $\Re(s) = 0$? To find out about possible zeros on this vertical line, consider the functional equation with $s = it, t \neq 0$. We obtain

$$\pi^{-it/2}\Gamma(it/2)\zeta(it) = \pi^{(it-1)/2}\Gamma\left(\frac{1-it}{2}\right)\zeta(1-it).$$

Since $\zeta(1-it) \neq 0$, it follows that $\zeta(it) \neq 0$. Therefore, the zeta function is zero-free on $\Re(s) = 0$.

So we are led to ask about the zeta zeros in the critical strip $0 < \Re(s) < 1$.

In the critical strip, the zeta zeros have the following two symmetries. By the functional equation, we have

$$\zeta\left(s\right) = 0 \Longrightarrow \zeta\left(1-s\right) = 0.$$

We already know that

$$\zeta\left(s\right) = 0 \Longrightarrow \zeta\left(\overline{s}\right) = 0$$

where $\overline{s}$ denotes the complex conjugate of $s \in \mathbb{C}$. So the zeros of the Riemann zeta function in the critical strip come in fours, that is, $s, \overline{s}, 1-s, 1-\overline{s}$, which lie on the vertices of a symmetrically placed rectangle in the complex plane: symmetrically placed about the critical line $\Re(s) = 1/2$. So the Riemann hypothesis says that the width of this rectangle is zero; in other words, the zeros all lie on the critical line.

## 3.3. Some zeta values

**Lemma 3.8.** $\zeta(0) = -1/2$.

**Proof.** We will use the recursive formula $\Gamma(s+1) = s\Gamma(s)$ and the fact that $\zeta(s)$ has a simple pole at $s = 1$ with residue 1. We start with the asymmetric form of the functional equation (3.6):

$$\zeta(s) = \zeta(1-s)2^s\pi^{s-1}\Gamma(1-s)\sin(\pi s/2).$$

Multiply by $1 - s$ to obtain

$$\begin{aligned}(1-s)\zeta(s) &= \zeta(1-s)2^s\pi^{s-1}(1-s)\Gamma(1-s)\sin(\pi s/2)\\ &= \zeta(1-s)2^s\pi^{s-1}\Gamma(2-s)\sin(\pi s/2).\end{aligned}$$

Now let $s \to 1$. We obtain

$$\begin{aligned}-1 &= \zeta(0)\cdot 2\cdot\Gamma(1)\cdot\sin\pi/2\\ &= 2\zeta(0)\end{aligned}$$

as required. ☐

**Lemma 3.9.** $\zeta(-1) = -\frac{1}{12}$.

**Proof.** To determine the value of $\zeta(-1)$, we use the functional equation. Setting $s = -1$, we have

$$\xi(-1) = \xi(2)$$

which gives

$$\sqrt{\pi}\Gamma\left(-1/2\right)\zeta\left(-1\right) = \pi^{-1}\Gamma\left(1\right)\zeta\left(2\right).$$

We also have

$$\begin{aligned}\Gamma\left(-1/2\right) &= \frac{1}{-1/2}\Gamma\left(1/2\right)\\ &= -2\Gamma\left(1/2\right)\\ &= -2\sqrt{\pi}\end{aligned}$$

and $\Gamma(1) = 1$. We infer that

$$\sqrt{\pi}\left(-2\right)\sqrt{\pi}\zeta\left(-1\right) = \frac{1}{\pi}\frac{\pi^2}{6}$$

so that

$$\zeta(-1) = -\frac{1}{12}.$$ ☐

**Lemma 3.10.** $\zeta(-3) = \frac{1}{120}$.

**Proof.** Substituting $s = -3$ in the functional equation

$$\zeta(s) = \zeta(1-s)2^s\pi^{s-1}\Gamma(1-s)\sin(\pi s/2),$$

we get

$$\begin{aligned}\zeta(-3) &= \zeta(4)2^{-3}\pi^{-4}\Gamma(4)\sin(-3\pi/2)\\ &= \frac{1}{90}\cdot\frac{1}{8}\cdot 3\cdot 2\cdot 1\\ &= \frac{1}{90}\cdot\frac{3}{4}\\ &= \frac{1}{120}.\end{aligned}$$

□

**Lemma 3.11.** $\zeta(1/2) \approx -1.46035$.

**Proof.** Here, the functional equation is of no use since $\zeta(1/2)$ occurs on both sides. However, we can proceed as follows. Define

$$\begin{aligned}\eta(s) &= 1 - 2^{-s} + 3^{-s} - 4^{-s} + \cdots\\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}.\end{aligned}$$

This series is convergent in $\Re(s) > 0$. Then we have

$$\begin{aligned}\eta(s) &= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots - \frac{2}{2^s}\left(1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots\right)\\ &= \zeta(s) - \frac{2}{2^s}\zeta(s)\\ &= (1-2^{1-s})\zeta(s).\end{aligned}$$

Setting $s = 1/2$, we have

$$\zeta(1/2) = \frac{1}{1-\sqrt{2}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}},$$

a slowly convergent series. □

Suppose that you were very naive and (like Ramanujan) knew no complex analysis and (like Ramanujan) had a unique and inexplicable mathematical intuition. You might stray from the path of rigor as follows. Moving outside the domain of definition of the zeta function, let $s = -1$ in the defining equation:

$$\begin{aligned}\zeta(-1) &= \sum_{n=1}^{\infty} \frac{1}{n^{-1}} \\ &= \sum_{n=1}^{\infty} n,\end{aligned}$$

leading to the spectacular equation

$$1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}.$$

This equation appeared in the first letter (received about January 31, 1913) from Ramanujan to Hardy.

This was Ramanujan's brave, and correct, evaluation of the zeta function at $s = -1$.

Here is another spectacular gem from the pen of Ramanujan (it appears in his first letter to Hardy [**W**, p. 167]):

$$1^3 + 2^3 + 3^3 + 4^3 + \cdots = 1/120.$$

How would you interpret this equation in terms of the zeta function?

## 3.4. Euler and the functional equation

We first need to reformulate the functional equation. Start with Corollary (3.6): for all $s \neq 1$,

$$\zeta(s) = \zeta(1-s) 2^s \pi^{s-1} \Gamma(1-s) \sin(\pi s/2).$$

Then

$$\begin{aligned}\zeta(1-s) &= (2\pi)^{-s} \pi \frac{1}{\Gamma(1-s)\sin(\pi s/2)} \zeta(s) \\ &= (2\pi)^{-s} \frac{\Gamma(s)\sin(\pi s)}{\sin(\pi s/2)} \\ &= 2(2\pi)^{-s} \cos(\pi s/2) \Gamma(s) \zeta(s)\end{aligned}$$

by the reflection principle [**MV**, C.6]:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

The functional equation of the zeta function, established by Riemann, who was the first to treat the zeta function as a function of the *complex variable* $s$, is given by

$$\zeta(1-s) = 2(2\pi)^{-s}\cos(\pi s/2)\Gamma(s)\zeta(s). \tag{3.6}$$

Euler was led to this equation 110 years before Riemann. Now Euler worked with real $s$. For $s$ a positive integer $\geq 2$, $1-s$ is a negative integer and so the series for the corresponding zeta value diverges, showing that its value has to be interpreted by a summation procedure; Euler used the generating function method (Abel summation). Euler worked with

$$\eta(s) = 1 - 2^{-s} + 3^{-s} - \cdots = (1-2^{1-s})\zeta(s)$$

rather than $\zeta(s)$ for better convergence and more accurate numerical evaluation.

Euler, in his paper [**Euler**, E352|1749], conjectured an equation which is equivalent to the functional equation of the zeta function. This work of Euler was forgotten, and 110 years later Riemann would rediscover this functional equation and provide a rigorous proof. All modern proofs of the functional equation involve mathematical tools which were unavailable to Euler, and it is remarkable that he was nevertheless able to predict its structure. Osler shows in his paper how Euler used divergent series to discover the functional equation for the zeta function [**Osler**].

In 1749 Euler gave a paper to the Berlin Academy entitled *Remarques sur un beau rapport entre les séries des puissances tant directs que réciproques*, [**Euler**, E352|1749].

Instead of using $\zeta(s)$, Euler uses the alternating series

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$$

which is defined in the wider region $\Re(s) > 0$. The function $\eta(s)$ is one simple step removed from $\zeta(s)$ as shown by the relation

$$\eta(s) = (1-2^{1-s})\zeta(s).$$

The functional equation (3.6) now becomes

$$(2^{s-1}-1)\eta(1-s) = -(2^s-1)\pi^{-s}\cos(\pi s/2)\Gamma(s)\eta(s). \tag{3.7}$$

Euler notes the following relations [**Euler**, E352|1749, §9]:

$$\begin{aligned}
\frac{1-2+3-4+5-6+\cdots}{1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\frac{1}{6^2}+\cdots} &= +\frac{1\cdot(2^2-1)}{(2-1)\pi^2},\\
\frac{1-2^2+3^2-4^2+5^2-6^2+\cdots}{1-\frac{1}{2^3}+\frac{1}{3^3}-\frac{1}{4^3}+\frac{1}{5^3}-\frac{1}{6^3}+\cdots} &= 0,\\
\frac{1-2^3+3^3-4^3+5^3-6^3+\cdots}{1-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\frac{1}{5^4}-\frac{1}{6^4}+\cdots} &= -\frac{1\cdot2\cdot3(2^4-1)}{(2^3-1)\pi^4},\\
\frac{1-2^4+3^4-4^4+5^4-6^4+\cdots}{1-\frac{1}{2^5}+\frac{1}{3^5}-\frac{1}{4^5}+\frac{1}{5^5}-\frac{1}{6^5}+\cdots} &= 0,\\
\frac{1-2^5+3^5-4^5+5^5-6^5+\cdots}{1-\frac{1}{2^6}+\frac{1}{3^6}-\frac{1}{4^6}+\frac{1}{5^6}-\frac{1}{6^6}+\cdots} &= \frac{1\cdot2\cdot3\cdot4\cdot5(2^6-1)}{(2^5-1)\pi^6},\\
\frac{1-2^6+3^6-4^6+5^6-6^6+\cdots}{1-\frac{1}{2^7}+\frac{1}{3^7}-\frac{1}{4^7}+\frac{1}{5^7}-\frac{1}{6^7}+\cdots} &= 0,\\
\frac{1-2^7+3^7-4^7+5^7-6^7+\cdots}{1-\frac{1}{2^8}+\frac{1}{3^8}-\frac{1}{4^8}+\frac{1}{5^8}-\frac{1}{6^8}+\cdots} &= -\frac{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7(2^8-1)}{(2^7-1)\pi^8},\\
\frac{1-2^8+3^8-4^8+5^8-6^8+\cdots}{1-\frac{1}{2^9}+\frac{1}{3^9}-\frac{1}{4^9}+\frac{1}{5^9}-\frac{1}{6^9}+\cdots} &= 0,\\
\frac{1-2^9+3^9-4^9+5^9-6^9+\cdots}{1-\frac{1}{2^{10}}+\frac{1}{3^{10}}-\frac{1}{4^{10}}+\frac{1}{5^{10}}-\frac{1}{6^{10}}+\cdots}& \\
= \frac{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9(2^{10}-1)}{(2^9-1)\pi^{10}}&
\end{aligned}$$

or if $n \geq 2$,

$$\begin{aligned}
\frac{\eta(1-n)}{\eta(n)} &= \frac{(-1)^{(n/2)+1}(2^n-1)(n-1)!}{(2^{n-1}-1)\pi^n} \quad \text{if} \quad n \quad \text{is even}\\
&= 0 \quad \text{if} \quad n \quad \text{is odd.}
\end{aligned}$$

These equations are now rewritten in the form

$$\frac{\eta(1-n)}{\eta(n)} = \frac{-(n-1)!(2^n-1)}{(2^{n-1}-1)\pi^n}\cdot\cos\frac{\pi n}{2}.$$

Euler, in his article [**Euler**, E352|1749, §10], now writes "je hazarderai la *conjecture suivante*, quelquefoit l'exposant $n$":

$$\frac{\eta(1-n)}{\eta(n)} = \frac{-1\cdot 2\cdot 3\cdots(n-1)(2^n-1)}{(2^{n-1}-1)\pi^n}\cdot\cos\frac{\pi n}{2}.$$

With $s = n$, this is precisely (3.7)!

Euler verified his conjecture for $n = 1/2$, which is

$$\frac{1-1/2^{1/2}+1/3^{1/2}-1/4^{1/2}+\cdots}{1-1/2^{1/2}+1/3^{1/2}-1/4^{1/2}+\cdots} = \frac{-\Gamma(1/2)(2^{1/2}-1)}{(2^{-1/2}-1)\pi^{1/2}}\cos\frac{\pi}{4}.$$

This is the only case considered by Euler in which both the series in the numerator and the denominator converge. He used $\Gamma(1/2) = \sqrt{\pi}$, and the right-hand side immediately reduces to 1. Euler remarked that his success in this case makes his conjecture very convincing. See [**Euler**, E352|1749, §14]:

"There should not be any further doubt about our conjecture, having verified it not only for all the cases where the variable $n$ is an integer, be it $n$ is positive, or $n$ is negative, but also for the case $n = 1/2$."

He goes on to test his conjecture for $n = (2k+1)/2$ with $k \in \mathbb{N}$. For $n = 3/2$ he obtains

$$\frac{1-\sqrt{2}+\sqrt{3}-\sqrt{4}+\sqrt{5}-\sqrt{6}+\cdots}{1-\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}-\frac{1}{4\sqrt{4}}+\frac{1}{5\sqrt{5}}-\frac{1}{6\sqrt{6}}+\cdots} = \frac{3+\sqrt{2}}{2\pi\sqrt{2}} = 0.4967738;$$

see [**Euler**, E352|1749, §15]. With $n = 15/2$ he gets

$$\frac{1-2^6\sqrt{2}+3^6\sqrt{3}-4^6\sqrt{4}+\cdots}{1-\frac{1}{2^7\sqrt{2}}+\frac{1}{3^7\sqrt{3}}-\frac{1}{4^7\sqrt{4}}+\cdots} = \frac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13(128\sqrt{2}-1)}{2^7(128-\sqrt{2})\pi^7};$$

see [**Euler**, E352|1749, §16].

## 3.5. The Euler constant revisited

Euler discovered the constant $\gamma$, defined by

$$\gamma := \lim_{n\to\infty}\left(\sum_{j=1}^{n}\frac{1}{j}-\log n\right).$$

This is a fundamental constant, now called Euler's constant. We give it to 50 decimal places as

$$\gamma = 0.57721\,56649\,01532\,86060\,65120\,90082\,40243\,10421\,59335\,93992\cdots.$$

There is a whole book devoted to the Euler constant, see Julian Havil [**H**].

The Euler constant $\gamma$ is equal to the constant term of $\zeta(s)$ at $s=1$, as we now proceed to show.

**Theorem 3.12.** *In the Laurent expansion of* $\zeta(s)$ *about* $s=1$, *we have*

$$\zeta(s)=\frac{1}{s-1}+\gamma+O\left(s-1\right).$$

**Proof.** First we note that

$$\begin{aligned}
\int_1^n \frac{[x]}{x^2} &= \int_1^2 \frac{dx}{x^2} + \int_2^3 \frac{2dx}{x^2} + \cdots + \int_{n-2}^{n-1}\frac{(n-2)dx}{x^2} + \int_{n-1}^{n}\frac{(n-1)dx}{x^2}\\
&= \left[-\frac{1}{x}\right]_1^2 + \left[-\frac{2}{x}\right]_2^3 + \cdots + \left[-\frac{n-2}{x}\right]_{n-2}^{n-1} + \left[\frac{n-1}{x}\right]_{n-1}^{n}\\
&= 1-\frac{1}{2}+2\left(\frac{1}{2}-\frac{1}{3}\right)+3\left(\frac{1}{3}-\frac{1}{4}\right)+4\left(\frac{1}{4}-\frac{1}{5}\right)+\cdots\\
&\quad +(n-2)\left(\frac{1}{n-2}-\frac{1}{n-1}\right)+(n-1)\left(\frac{1}{n-1}-\frac{1}{n}\right)\\
&= 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n-1}-\frac{n-1}{n}\\
&= \frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}.
\end{aligned}$$

Now we apply Theorem 1.10. We obtain

$$\begin{aligned}\lim_{s\to 1}\left(\zeta(s)-\frac{1}{s-1}\right) &= 1-\int_1^\infty \frac{\{x\}}{x^2}dx\\ &= 1+\lim_{n\to\infty}\int_1^n \frac{[x]-x}{x^2}dx\\ &= 1+\lim_{n\to\infty}\left(\int_1^n \frac{[x]dx}{x^2}-\log n\right)\\ &= 1+\lim_{n\to\infty}\left(\sum_{j=2}^n \frac{1}{j}-\log n\right)\\ &= \lim_{n\to\infty}\left(\sum_{j=1}^n \frac{1}{j}-\log n\right)\\ &= \gamma. \end{aligned}$$ □

In the next result, we expand the gamma function $\Gamma(s)$ as a Taylor series about $s = 1$ and show that the coefficient of the linear term is $-\gamma$.

**Theorem 3.13.** *We have*

$$\Gamma(s) = 1 - \gamma(s-1) + O(s-1)^2.$$

**Proof.** This is the Taylor series expansion for $\Gamma(s)$ around $s = 1$:

$$\Gamma(s) = \Gamma(1) + \Gamma'(1)(s-1) + O(s-1)^2.$$

We have $\Gamma(1) = 1$. In addition, we need Euler's formula

$$\Gamma'(1) = -\gamma;$$

see [**Euler**, E368|1765 §14]. To prove Euler's formula, we can proceed as follows. Introduce the *digamma function*

$$\psi(s) := \frac{\Gamma'(s)}{\Gamma(s)} = \frac{d}{ds}\log\Gamma(s).$$

The digamma function is meromorphic and satisfies the difference equation

$$\psi(s+1) = \psi(s) + \frac{1}{s}$$

which is inherited from the recurrence relation $\Gamma(s+1) = s\Gamma(s)$. A simple iteration now yields the equation

$$\psi(x) = \psi(x+n) - \sum_{j=0}^{n-1} \frac{1}{x+j}$$

from which follows

$$\psi(x) = \lim_{n\to\infty} \left( \log n - \sum_{j=0}^{n-1} \frac{1}{x+j} \right).$$

This limit is valid for all $x \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, using the fact that $\psi(x) - \log x$ goes unformly to 0 in the right half-plane as $\Re(x) \to \infty$; see [**L**, 3.1]. This formula gives $\psi(1) = -\gamma$ as required. ☐

The Euler constant is inextricably entwined with the zeta function and the gamma function. Here's another example, which relates $\gamma$ to the zeta values at the natural numbers greater than or equal to 2. The result is due to Euler [**Euler**, E583|1776, §6].

**Theorem 3.14.**

$$1 - \gamma = \sum_{n=2}^{\infty} \frac{\zeta(n) - 1}{n}.$$

**Proof.** Recall the definition of the Euler constant $\gamma$ which was given by

$$\gamma = \lim_{n\to\infty} \left( \sum_{m=1}^{n} \frac{1}{m} - \log n \right).$$

We therefore have

$$\begin{aligned}
\gamma &= \lim_{n\to\infty}\left(\sum_{m=1}^{n}\frac{1}{m}-\sum_{m=2}^{n}\log\left(\frac{m}{m-1}\right)\right)\\
&= 1+\sum_{m=2}^{\infty}\left(\frac{1}{m}+\log\left(\frac{m-1}{m}\right)\right)\\
&= 1+\sum_{m=2}^{\infty}\left(\frac{1}{m}+\log\left(1-\frac{1}{m}\right)\right)\\
&= 1-\sum_{m=2}^{\infty}\sum_{k=2}^{\infty}\frac{1}{km^k}\\
&= 1-\sum_{k=2}^{\infty}\sum_{m=2}^{\infty}\frac{1}{km^k}\\
&= 1-\sum_{k=2}^{\infty}\frac{1}{k}\sum_{m=2}^{\infty}\frac{1}{m^k}\\
&= 1-\sum_{k=2}^{\infty}\frac{\zeta(k)-1}{k},
\end{aligned}$$

which is the required result. □

Select a prime number $p$. Given $x \in \mathbb{Q}$, we have its unique factorization as powers of prime numbers, $x = p^k \cdot p_1^{k_1} \cdots p_n^{k_n}$ where $p, p_1, \ldots, p_n$ are distinct primes and $k, k_1, k_2, \ldots, k_n \in \mathbb{Z}$. The $p$-adic norm is defined as follows:

$$x = p^k \cdot p_1^{k_1} \cdots p_n^{k_n} \implies |x|_p = p^{-k}.$$

The $p$-adic norm makes a beeline for the exponent of $p$ and then reverses it. The completion of $\mathbb{Q}$ with respect to the norm $|\cdot|_p$ is $\mathbb{Q}_p$, the field of $p$-adic numbers.

We also have the absolute value $|x|$ which will presently be denoted $|x|_\infty$. The completion of $\mathbb{Q}$ with respect to the absolute value $|\cdot|_\infty$ is the real field $\mathbb{R}$.

We recall that a *place* of $\mathbb{Q}$ is a completion of $\mathbb{Q}$. We have the finite places $\mathbb{Q}_p$ and the infinite place $\mathbb{R}$.

Euler's constant $\gamma$ appears in the Laurent expansion of the Riemann zeta function at $s = 1$; here the zeta function is a function encoding the properties of the finite primes. Euler's constant also appears in the Taylor expansion of the gamma function at $s = 1$; here the gamma function encodes properties of the infinite place.

The Euler constant occurs in two apparently different contexts: one associated to the finite places of $\mathbb{Q}$ and the other associated to the infinite place of $\mathbb{Q}$. There is a well-known connection between these two contexts, given by the product formula for $\mathbb{Q}$ over all places, finite and infinite:

$$\prod_{2\le p\le\infty} |x|_p = 1.$$

The functional equation can be written

$$\zeta^*(s) = \zeta^*(1-s)$$

where

$$\begin{aligned}\zeta^*(s) :&= \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)\\ &= \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\prod_p\left(1-\frac{1}{p^s}\right)^{-1}.\end{aligned}$$

If we write

$$\begin{aligned}\zeta_p &= (1-p^{-s})^{-1},\\ \zeta_\infty &= \pi^{-s/2}\Gamma\left(\frac{s}{2}\right),\end{aligned}$$

then we have

$$\prod_{2\le p\le\infty}\zeta_p = \zeta^*.$$

This point of view is the point of view of Tate's thesis [**Tate**], and indeed of Weil's book [**We2**].

**3.5.1. The Euler constant and periods.** This section is inspired by the article of Kontsevich and Zagier [**KZ**] and by §3.14 in the article by Lagarias [**L**]. Numbers which are not algebraic are called **transcendental**. There is a huge difference in size between algebraic and transcendental numbers: the set $\overline{\mathbb{Q}}$ of algebraic numbers is *countable* and the set of transcendental numbers is *uncountable*.

There is, however, one further important class of numbers, lying between $\overline{\mathbb{Q}}$ and $\mathbb{C}$, which is missing in the above classification. This "new" class of numbers, the **periods**, seems to be the next most important class in the hierarchy of numbers according to their arithmetic properties. The periods form a countable class. Periods appear surprisingly often in various formulas and conjectures in mathematics.

**Definition 3.15** ([**KZ**, §1.1])**.** A *period* is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with coefficients in $\mathbb{Q}$, over domains in $\mathbb{R}^n$ given by polynomial inequalities with coefficients in $\mathbb{Q}$.

We will denote the set of periods by $\mathcal{P}$. It is obviously countable. Sums and products of periods remain periods, so the periods form a ring. Maxim Kontsevich and Don Zagier [**KZ**] gave a survey of periods and introduced some conjectures about them.

The irrational number $\sqrt{2}$ is a period, for we have

$$\sqrt{2} = \int_{2x^2 \leq 1} dx.$$

The periods are intended to bridge the gap between the algebraic numbers and the transcendental numbers. The class of algebraic numbers is too narrow to include many common mathematical constants, while the set of transcendental numbers is not countable. Periods are usually transcendental numbers, but they are described by, and contain, only a finite amount of information and (at least conjecturally) can be identified in an algorithmic way.

The set of all periods forms a ring $\mathcal{P}$ which includes the field $\overline{\mathbb{Q}}$ of all algebraic numbers. It follows that if Euler's constant is not a period, then it must be a transcendental number. We can express $\pi$ as a period by any of the following integrals:

$$\pi = \iint\limits_{x^2+y^2 \leq 1} dxdy = 2\int_{-1}^{1} \sqrt{1-x^2}dx = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

or also, after multiplication by the algebraic number $2i$, by the contour integral

$$2\pi i = \oint \frac{dz}{z}$$

in the complex plane around the point 0.

Two other famous numbers which have special notations are

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.7182818\ldots,$$

the basis of natural logarithms, and Euler's constant

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n\right) = 0.5772156\ldots$$

but these two numbers (conjecturally) are not periods. It is known only that $e$ is transcendental.

However, there are many examples of periods besides $\pi$ and the algebraic numbers. For example, logarithms of algebraic numbers are periods; e.g.,

$$\log 2 = \int_1^2 \frac{dx}{x}.$$

Many infinite sums of elementary expressions are periods. For example

$$\zeta(3) = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots = 1.2020569\ldots$$

has the representation

$$\zeta(3) = \iiint\limits_{0<x<y<z<1} \frac{dxdydz}{(1-x)yz}$$

as an integral, and more generally, all values of the Riemann zeta function

$$\zeta(s) = \sum_{n\geq 1}\frac{1}{n^s}$$

at integers $s \geq 2$ are periods, as are the "*multiple zeta values*"

$$\zeta(s_1,\ldots,s_k) := \sum_{0<n_1<\cdots<n_k}\frac{1}{n_1^{s_1}\cdots n_k^{s_k}}$$

with $s_i \in \mathbb{N}, s_k \geq 2$.

Values of the gamma function

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt$$

at rational values of the argument $s$ are closely related to periods:

$$\Gamma(p/q)^q \in \mathcal{P}$$

with $p, q \in \mathbb{N}$. For instance,

$$\Gamma(1/2)^2 = \pi$$

and

$$\Gamma(1/3)^3 = 2^{4/3} 3^{1/2} \pi \int_0^1 \frac{dx}{\sqrt{1-x^3}}.$$

We have, in this book, met $\zeta(2)$ on several occasions. Here is a realization of $\zeta(2)$ as a period over $\mathbb{Q}$; see [**HMS**, Proposition 12.1.7]:

$$\int_{0 \le x \le y \le 1} \frac{dxdy}{(1-x)y} = \zeta(2).$$

In 2001, Kontsevich and Zagier posed the following problem [**KZ**]: *exhibit at least one number which does not belong to* $\mathcal{P}$.

Of course such numbers exist since $\mathcal{P}$ is countable. Solving this open problem would be the analogue of Liouville's achievement in the 19th century when he constructed the first explicit example of a number which could be proved to be transcendental. Even more desirable would be to emulate the achievements of Hermite and Lindemann and prove that some specific numbers of interest, like $e$ or $1/\pi$, do not belong to $\mathcal{P}$.

In 2008, M. Yoshinaga wrote down a nonperiod

$$\alpha = 0.77766444\ldots.$$

This is a computable real number which is not a period. For a recent account of this result, see [**HMS**, §16.6].

There are many famous unsolved problems about the nature of the Euler constant. The most well-known one is

**Conjecture 3.16.** *Euler's constant is irrational.*

This is a longstanding open problem. A recent and stronger version of it is the following.

**Conjecture 3.17.** *Euler's constant is not a period. In particular, Euler's constant is transcendental.*

**Definition 3.18.** An *exponential period* is an absolutely convergent integral of the product of an algebraic function with the exponent of an algebraic function, over a real semialgebraic set, where all polynomials entering the definition have algebraic coefficients.

Here is the classic example of an exponential period:

$$\sqrt{\pi} = \int_{-\infty}^{+\infty} \exp(-x^2)dx.$$

Recall the definition (3.2) of the gamma function for $\Re(s) > 1$:

$$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt.$$

With a real variable $x$ this becomes

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$$

which is precisely Euler's formula [**Euler**, E368|1769 §11].

Differentiation under the integral sign yields

$$\Gamma'(1) = \int_0^\infty e^{-x}\log x\, dx.$$

Combining this with Euler's result [**Euler**, E368|1769 §14]

$$\Gamma'(1) = -\gamma,$$

we have

$$-\gamma = \int_0^\infty e^{-x}\log x\, dx.$$

Substituting $-\log x = \int_x^1 y^{-1}dy$ for $0 < x < 1$ and

$$\log x = \int_1^x y^{-1}dy$$

for $x > 1$ yields the formula (due to Kontsevich):

$$\gamma = \int_0^1\int_x^1 \frac{e^{-x}}{y}dydx - \int_1^\infty\int_1^x \frac{e^{-x}}{y}dydx.$$

This elegant formula certifies that $\gamma$ is an exponential period.

## Chapter 4

# The explicit formulas in analytic number theory

In the present context, an *explicit formula* is an equation of the following form:

$$\begin{aligned} &\textit{a counting function depending only on the prime numbers}\\ &\quad = \textit{a formula depending only on the zeta zeros.}\end{aligned}$$

Such an explicit formula is said to exhibit the reciprocity between the prime numbers and the zeta zeros.

The first such explicit formula was discovered by Riemann in his memoir of 1859 [**R**]. Later, von Mangoldt proved his own version of the Riemann formula. It is customary to focus attention on the von Mangoldt formula, but in this book the focus will be on the Riemann formula, which deals explicitly with the difference $\pi(x) - \mathrm{Li}(x)$.

On the right-hand side of an explicit formula we have the set of zeta zeros. For reasons which will become clear in this chapter, the imaginary parts of the nontrivial zeta zeros are of particular importance. We recall that the zeta zeros occur in complex conjugate pairs. For those zeta zeros $\rho$ with $\Im(\rho) > 0$, it is customary to write the imaginary parts as $\gamma_1 < \gamma_2 < \gamma_3 < \cdots$.

In the book by Mazur and Stein [**MS**], the set $\{\gamma_1, \gamma_2, \gamma_3, \ldots\}$ is called the Riemann spectrum of the primes.

In this chapter, we will state and prove the two explicit formulas and discuss their relationship with the Riemann hypothesis. The Riemann spectrum may be interpreted as a sound spectrum, and we investigate how this sound varies with time.

## 4.1. The von Mangoldt explicit formula

We recall Definition 2.5 of the function $\Lambda(n)$ where $n \in \mathbb{Z}$. We have $\Lambda(n) = \log p$ if $n = p^k$ and $\Lambda(n) = 0$ otherwise, where $p$ is a prime number. Recall also the logarithmic derivative of $\zeta(s)$:

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

We introduce the new function

$$\psi(x) := \frac{1}{2}\left(\sum_{n<x} \Lambda(n) + \sum_{n \leq x} \Lambda(n)\right).$$

**Theorem 4.1.** *The von Mangoldt explicit formula is given by*

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right) - \log 2\pi \tag{4.1}$$

*where* $\rho$ *is the zeta zeros which lie in the critical strip* $0 < \Re(s) < 1$.

Note that the left-hand side of the von Mangoldt explicit formula (4.1) depends only upon the prime numbers, and the right-hand side depends only on the nontrivial zeros $\rho$ of the Riemann zeta function. We can, if we so choose, incorporate the trivial zeros into this formula in the following way:

$$\begin{aligned} -\log\left(1 - \frac{1}{x^2}\right) &= \sum_{n=1}^{\infty} \frac{x^{-n}}{n} \\ \implies -\frac{1}{2}\log\left(1 - \frac{1}{x^2}\right) &= \sum_{n=1}^{\infty} \frac{x^{-2n}}{2n} \\ &= -\sum_{n=1}^{\infty} \frac{x^{-2n}}{-2n}. \end{aligned}$$

The integers $\{-2n : n = 1, 2, 3, \ldots\}$ are the trivial zeros of the Riemann zeta function. So we can now write the von Mangoldt explicit formula in the following way:

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \log(2\pi)$$

or

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} + O(1)$$

where the summation is now over *all* the zeta zeros.

At this point, we are strongly influenced by §10.2 in Jeffrey Stopple's book [**St**]. This reference contains some excellent diagrams, especially Figure 10.2.

The issue for us at the moment is to seek a simple geometrical interpretation of the von Mangoldt formula. Recall that the nontrivial zeta zeros occur in complex conjugate pairs $\rho$ and $\overline{\rho}$. If $\rho = \beta + i\gamma$, then noting that

$$x^{i\gamma} = e^{i\gamma \log x} \qquad \text{and} \qquad |\rho|^2 = \beta^2 + \gamma^2,$$

we have

$$\begin{aligned}
\frac{x^{\rho}}{\rho} + \frac{x^{\overline{\rho}}}{\overline{\rho}} &= \frac{x^{\beta+i\gamma}}{\beta+i\gamma} + \frac{x^{\beta-i\gamma}}{\beta-i\gamma} \\
&= x^{\beta} \cdot \frac{(\beta-i\gamma)x^{i\gamma} + (\beta+i\gamma)x^{-i\gamma}}{\beta^2+\gamma^2} \\
&= \frac{x^{\beta}}{\beta^2+\gamma^2}\left(\beta(x^{i\gamma}+x^{-i\gamma}) - i\gamma(x^{i\gamma}-x^{-i\gamma})\right) \\
&= \frac{2x^{\beta}}{\beta^2+\gamma^2}\left(\beta\cos(\gamma\log x) + \gamma\sin(\gamma\log x)\right) \\
&= \frac{2x^{\beta}}{|\rho|}\left(\frac{\beta}{|\rho|}\cos(\gamma\log x) + \frac{\gamma}{|\rho|}\sin(\gamma\log x)\right) \\
&= \frac{2x^{\beta}}{|\rho|}\cos(\gamma\log x + \theta)
\end{aligned}$$

where $\theta$ is such that $\cos\theta = \beta/|\rho|$ and $-\sin\theta = \gamma/|\rho|$.

We could use a logarithmic scale. Let $t = \log x$; that is, $x = e^t$. Then we have

$$\frac{x^\rho}{\rho} + \frac{x^{\overline{\rho}}}{\overline{\rho}} = \frac{2e^{\beta t}}{|\rho|} \cdot \cos(\gamma t + \theta)\,.$$

This can be interpreted as an *oscillatory* term, with $t$ as time variable:

– the **amplitude** is $2e^{\beta t}/|\rho|$;

– the **phase shift** is $\theta$;

– the **period** is $2\pi/\gamma$;

– the **frequency** is $\gamma$.

*Each complex conjugate pair of nontrivial zeta zeros contributes an oscillatory term, with a definite amplitude, phase shift, period, and frequency.*

For those zeta zeros $\rho$ in the critical strip with $\Im(\rho) > 0$, we will denote the imaginary parts by $\gamma_1 < \gamma_2 < \gamma_3 < \cdots < \gamma_n < \cdots$. So we have $\rho_n = \beta_n + i\gamma_n$. It is then clear that as $n \to \infty$ we have

– the $n$th amplitude $2e^{\beta_n t}/|\rho_n| \to 0$;

– the $n$th phase shift $\theta_n = \cos^{-1}(\beta_n/|\rho_n|) \to \pi/2$;

– the $n$th frequency $\gamma_n \to \infty$.

We list the first few nontrivial zeros of the zeta function:

$$\begin{gathered}
\rho_1 \approx 1/2 + (14.12473)\,i, \\
\rho_2 \approx 1/2 + (21.02204)\,i, \\
\rho_3 \approx 1/2 + (30.42488)\,i, \\
\vdots \\
\rho_{100} \approx 1/2 + (236.52423)\,i, \\
\vdots
\end{gathered}$$

*The case when* $\beta = 1/2$. Suppose for the moment that the Riemann hypothesis is true. Then all the nontrivial zeta zeros lie on the line $\Re(s) = 1/2$. In other words, we have that $\beta = 1/2$. So we can now write

$$\frac{x^{\rho}}{\rho} + \frac{x^{\overline{\rho}}}{\overline{\rho}} = \frac{2x^{1/2}}{|\rho|} \cos(\gamma \log x + \theta).$$

The envelope of this curve is then the parabola $y = \pm 2x^{1/2} / |\rho|$ or

$$y^2 = 4x / |\rho|^2 .$$

As $|\rho|$ increases, we can see that the parabola $y^2 = 4x / |\rho|^2$ gets closer and closer to the $x$-axis.

For a vivid graphical display of the von Mangoldt formula, go to `www.demonstrations.wolfram.com` and enter *Using Zeta Zeros to Compute the Chebyshev Psi Function* in the search tab. If you download the *Wolfram Player* app, then you can vary the parameters. Even with the first 200 pairs of zeta zeros, you can see how well the von Mangoldt formula hugs the step function $\psi(x)$.

## 4.2. Can you hear the Riemann hypothesis?

We consider once again the von Mangoldt explicit formula

$$\psi(x) = x - \sum_{\zeta(\rho)=0} \frac{x^{\rho}}{\rho} + O(1) .$$

The left-hand side of this equation is a step function. On the right-hand side, we have already shown that each complex conjugate pair of zeta zeros gives rise to an oscillatory term, with amplitude $2e^{\beta t} / |\rho|$. If we fix a zeta zero $\rho$, then this amplitude is increasing as $t$ increases. If we think of ourselves as listening to a sound wave of this amplitude, then we will experience a *crescendo*; i.e., the sound will get louder. We attempt to quantify this phenomenon. An $\alpha$-*crescendo* is the ratio $\alpha$ of the amplitudes taken over a time interval of one second. With initial time $T$, the amplitude ratio is

$$\left( \frac{2e^{\beta(T+1)}}{|\rho|} \right) / \left( \frac{2e^{\beta T}}{|\rho|} \right) = \left( \frac{2e^{\beta(T+1)}}{|\rho|} \right) \left( \frac{|\rho|}{2e^{\beta T}} \right) = e^{\beta}$$

which is independent of the initial time $T$. Each conjugate pair of nontrivial zeta zeros gives rise to a pure tone with an $e^{\beta}$-crescendo. So if the Riemann hypothesis is true, then each conjugate pair of zeta zeros gives rise to a pure note with a $\sqrt{e}$-crescendo, and as $t$ increases the loudness of this note increases at a constant rate. When we take the superposition of all the zeta zeros, we obtain a *chord.* We introduce the term *chord of the zeta zeros*. The first nontrivial zeta zero gives us the fundamental frequency of this chord; the harmonics are given by the other zeros.

Consider NH (Negation of the Riemann hypothesis). In particular suppose that there exists a single zeta zero $s_0$ which is off the critical line:

$$s_0 = \beta_0 + i\gamma_0, \qquad \beta_0 \neq 1/2.$$

Let $\beta_0 = \frac{1}{2} + \delta$ with $\delta \neq 0$. For this pure note, we have a $\sqrt{e} \cdot e^{\delta}$-crescendo.

If $\delta > 0$, then this pure tone will get louder *more rapidly* than all the other pure tones in the chord of the zeta zeros. A listener with very acute hearing should be able to detect this!

If, on the other hand, $\delta < 0$, then this pure tone will get louder *more slowly* than all the other pure tones in the chord of the zeta zeros. All the other tones are performing a crescendo at exactly the same rate $\sqrt{e}$.

*On the relation between loudness and amplitude.* We often express amplitudes in logarithmic units called *decibels.* If $\alpha$ is the amplitude of a signal, then we can define the decibel (dB) level $d$ as

$$\begin{aligned} d &= 20 \cdot \log_{10} \alpha/\alpha_0 \\ &= 20 \cdot \log_{10} \alpha - 20 \cdot \log_{10} \alpha_0 \end{aligned}$$

where $\alpha_0$ is a reference amplitude. A convenient choice of $\alpha_0$ is

$$\alpha_0 = 10^{-5}.$$

When the amplitude $\alpha$ is multiplied, as in the present context, by $e^{\beta}$, the decibel level rises by

$$20 \cdot \log_{10} e^{\beta} = 20 \cdot \beta \log_{10} e \approx 20\beta(0.4343) = 8.686\beta.$$

When $\beta = 1/2$, the rise in decibel level is $\approx 4.343$. Amplitude is related in an inexact way to the perceived loudness of sound. But amplifying a signal by 3 dB, say, will fairly reliably make it sound about one “step” louder. So, in our case, amplifying the signal by 4.343 will certainly be audible. For more information on the science of sound, in particular the perception of sound, see [**RMW**, p. 99].

*If the Riemann hypothesis is true, then we will obtain a steady crescendo for each pure tone in the chord of the zeta zeros: for each unit of time, the dB level will rise by by* 4.343*.*

## 4.3. Comparison with Fourier series

We turn once again to the von Mangoldt formula:

$$\psi\left(x\right) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} + O\left(1\right)$$

where the summation is over all the zeta zeros. If the Riemann hypothesis is true, then we can write

$$\begin{aligned} \psi\left(x\right) &= x - \sum_{\rho} \frac{x^{1/2+i\gamma}}{\rho} + O\left(1\right) \\ \implies \frac{\psi\left(x\right) - x}{\sqrt{x}} &= -\sum_{\rho} \frac{x^{i\gamma}}{\rho} + O(1). \end{aligned}$$

We know from above that

$$\frac{x^{\rho}}{\rho} + \frac{x^{\overline{\rho}}}{\overline{\rho}} = \frac{2x^{\rho}}{\beta^2+\gamma^2}\left(\beta\cos\left(\gamma\log x\right) + \gamma\sin\left(\gamma\log x\right)\right).$$

Now

$$\begin{aligned} \sum_{\gamma>0} \sin(\gamma\log x)\left(\frac{1}{\gamma} - \frac{\gamma}{\beta^2+\gamma^2}\right) &= \sum_{\gamma>0} \sin\left(\gamma\log x\right)\frac{\beta^2+\gamma^2-\gamma^2}{\gamma\left(\beta^2+\gamma^2\right)} \\ &= \sum_{\gamma>0} \sin\left(\gamma\log x\right)\left(\frac{\beta^2}{\gamma\left(\beta^2+\gamma^2\right)}\right) \\ &= O\left(1\right) \end{aligned}$$

since $\beta^2 < 1$ and $\sum_{\gamma>0} 1/\gamma^2 < \infty$. Similarly, we have

$$\sum_{\gamma>0} \frac{\beta\cos\left(\gamma\log x\right)}{\beta^2+\gamma^2} = O\left(1\right).$$

We infer that

$$\frac{\psi\left(x\right) - x}{\sqrt{x}} = -2\sum_{\gamma>0} \frac{\sin\left(\gamma\log x\right)}{\gamma} + O\left(1\right).$$

If, as before, we write the zeta zeros $\rho$ in the critical strip with $\Im(\rho) > 0$ as $\frac{1}{2}+i\gamma_1, \frac{1}{2}+i\gamma_2, \frac{1}{2}+i\gamma_3, \ldots$ with $0 < \gamma_1 < \gamma_2 < \gamma_3 < \cdots$, then we have

$$\frac{\psi(x)-x}{\sqrt{x}} = -2\sum_{n=1}^{\infty}\frac{\sin(\gamma_n \log x)}{\gamma_n} + O(1).$$

If we set $t = \log x$, then this may be written

$$\frac{\psi(x)-x}{\sqrt{x}} = -2\sum_{n=1}^{\infty}\frac{\sin(\gamma_n t)}{\gamma_n} + O(1). \tag{4.2}$$

Now we have the following standard Fourier series:

$$\frac{2x-1}{2} = -2\sum_{n=1}^{\infty}\frac{\sin(2\pi n x)}{2\pi n}$$

with $x \in [0,1]$. We may extend the domain of $x$ to the whole real line as follows:

$$\frac{2x-2[x]-1}{2} = -2\sum_{n=1}^{\infty}\frac{\sin(2\pi n x)}{2\pi n}. \tag{4.3}$$

At this point, it is interesting to compare (4.3) with (4.2). In the case of (4.3) we have the frequencies $2\pi, 4\pi, 6\pi, \ldots$. These frequencies are all integer multiples of a single frequency (the fundamental frequency). In the case of (4.2) we have the frequencies $\gamma_1, \gamma_2, \gamma_3, \ldots$. Here the situation is quite different. It is conjectured that these frequencies are all linearly independent.

Even so, the superficial similarity of (4.3) and (4.2) has led us to interpret the right-hand side of (4.2) as a sound wave, a superposition of pure tones. Such a sound file can be made with the *Mathematica* package called *Audio*.

*Historical comment.* The equation (4.2) was the starting point in Littlewood's proof of his famous theorem concerning the sign oscillation of $\pi(x) - \mathrm{Li}(x)$. This equation appears as equation (5) in his very terse Comptes Rendus Note [**Li**], published shortly before the outbreak of WW1. He writes $\eta = \log x$ so his equation (5) appears as

$$\frac{\psi(x)-x}{\sqrt{x}} = -2\sum_{1}^{\infty}\frac{\sin\gamma_n\eta}{\gamma_n} + O(1).$$

## 4.4. Proof of the von Mangoldt formula

Where does the von Mangoldt formula come from? Recall the definition of the Euler product, which was defined in Chapter 2 as

$$\prod_p \left(1 - p^{-s}\right)^{-1}.$$

This is a product over primes. There is a totally different way of thinking of the zeta function, which is to think of it as a polynomial of infinite degree. In that case, leaving aside its simple pole at $s = 1$, it should be possible to split it into its linear factors. This heuristic argument at least makes it plausible that, for $\Re(s) > 0$, the correct infinite product (the Hadamard product) is

$$\frac{1}{2\,(s-1)} \left(\frac{2\pi}{e}\right)^s \prod_\rho (1 - s/\rho) e^{s/\rho}$$

where the product is taken over all the zeta zeros $\rho$. You can see the simple pole at $s = 1$ and each zero at $s = \rho$.

We have already met the Weierstrass factorization theorem in Section 3.1. It's time to state it explicitly [**Des**, p. 220].

**Theorem 4.2.** *Let $f$ be an entire function with $\{z_1, z_2, \ldots, z_j, \ldots\}$ as its zero set, each $z_j$ being counted as often as its multiplicity. Let $m$ be the multiplicity of $0$ ($m$ may be $0$). Then there exist integers $p_1, p_2, \ldots, p_j, \ldots$ and an entire function $g$ such that*

$$f(z) = \exp(g(z)) \cdot z^m \cdot \prod_{j=1}^{\infty} E_{p_j}(z/z_j)$$

*where*

$$\begin{aligned} E_0(z) &= 1 - z, \\ E_1(z) &= (1 - z) \exp z, \\ &\ \ \vdots \\ E_p(z) &= (1 - z) \exp(z + z^2/2 + \cdots + z^p/p). \end{aligned}$$

The Hadamard product fits into this theorem by taking

$$\begin{aligned} f(s) &= (s-1)^{-1}\zeta(s), && \Re(s) > 0, \\ m &= 0, \\ g(s) &= s\log(2\pi/e) - \log 2, \\ p_j &= 1, \end{aligned}$$

for all $j = 1, 2, 3, \ldots$.

The next crucial step is to equate these two infinite products for $\zeta(s)$:

$$\prod_p \left(1 - p^{-s}\right)^{-1} = \frac{1}{2\,(s-1)} \left(\frac{2\pi}{e}\right)^s \prod_\rho (1 - s/\rho)\, e^{s/\rho}. \tag{4.4}$$

Note that all the primes are on the left-hand side and all the zeta zeros are on the right-hand side. The explicit formulas in number theory are all obtained from (4.4) by various clever manipulations. In the course of these manipulations, the *duality* between primes and zeta zeros is preserved.

We will now multiply through by the pole at $s = 1$:

$$(s-1)\,\zeta\,(s) = \frac{1}{2} \left(\frac{2\pi}{e}\right)^s \prod_\rho (1 - s/\rho)\, e^{s/\rho}. \tag{4.5}$$

We now take the logarithmic derivative of (4.5); i.e., we first take the logarithm of each side and then differentiate. Spelling out the details, we get, for the left-hand side,

$$\frac{\zeta(s) + (s-1)\zeta'(s))}{(s-1)\zeta(s)} = \frac{1}{s-1} + \frac{\zeta'(s)}{\zeta(s)}.$$

If $f\,(s) = (1 - s/\rho)\, e^{s/\rho}$, then

$$\begin{aligned} \frac{f'(s)}{f(s)} &= \frac{(-1/\rho)\, e^{s/\rho} + (1 - s/\rho)\, e^{s/\rho}/\rho}{(1 - s/\rho)\, e^{s/\rho}} \\ &= \frac{-s/\rho^2}{1 - s/\rho} \\ &= \frac{s}{\rho\,(s-\rho)}. \end{aligned}$$

If $f(s) = \frac{1}{2}\left(\frac{2\pi}{e}\right)^s$, then we get

$$\frac{f'(s)}{f\,(s)} = \log(2\pi) - 1.$$

Putting all of this together, we have that the logarithmic derivative of the Hadamard product is

$$\frac{1}{s-1} + \frac{\zeta'(s)}{\zeta(s)} = \sum_{\rho} \frac{s}{\rho\,(s-\rho)} + \log\,(2\pi) - 1.$$

Then we we have

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{s}{1-s} + \sum_{\rho} \frac{s}{\rho\,(s-\rho)} + \log\,(2\pi)\,.$$

By Theorem 2.6, we have

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \frac{\Lambda\,(n)}{n^s}. \tag{4.6}$$

We have

$$-\sum_{n=1}^{\infty} \frac{\Lambda\,(n)}{n^s} = \frac{s}{1-s} + \sum_{\rho} \frac{s}{\rho\,(s-\rho)} + \log\,(2\pi)\,.$$

Note that the fundamental duality between primes and zeta zeros is preserved in this equation: the left-hand side depends entirely on the primes, and the right-hand side depends entirely on the zeta zeros.

We get to work on (4.6):

$$\begin{aligned}
-\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} &= -\sum_{n=1}^{\infty} \frac{\psi(n+1)-\psi(n)}{n^s} \\
&= -\sum_{n=1}^{\infty} \psi(n) \left( \frac{1}{n^s} - \frac{1}{(n+1^s)} \right) \\
&= -\sum_{n=1}^{\infty} \psi(n) \int_n^{n+1} x^{-s-1} dx \\
&= -\sum_{n=1}^{\infty} \int_n^{n+1} \psi(x) x^{-s-1} dx \\
&= -s \int_1^{\infty} \psi(x) x^{-s-1} dx \\
&= -s (\mathcal{M}\psi)(s)
\end{aligned}$$

where $\mathcal{M}$ denotes the Mellin integral transform, which is defined as follows.

**Definition 4.3.** The Mellin transform $(\mathcal{M}f)(s)$ is defined as

$$(\mathcal{M}f)(s) = \int_1^{\infty} f(x) x^{-s-1} dx$$

under certain conditions on the function $f$ and the variable $s$.

The Mellin transform has the following properties: $\mathcal{M}$ is linear and injective.

We now take the Mellin transform of the right-hand side of the von Mangoldt explicit formula. That is, we take the Mellin transform of

$$x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \log(2\pi).$$

We require an elementary property of the Mellin transform. Namely,

$$(\mathcal{M}(x^a))(s) = \frac{1}{s-a}$$

for $\Re(s) > \Re(a)$; see Exercise 5.3 in the final section of this book. Substituting $a = 1, \rho, 0$ in this formula, we get

$$\begin{aligned} -s\left(\mathcal{M}\left(x\right)\right)\left(s\right) &= \frac{s}{1-s}, \\ -s\mathcal{M}\left(\frac{-x^{\rho}}{\rho}\right)\left(s\right) &= \frac{s}{\rho\left(s-\rho\right)}, \\ -s\mathcal{M}(-\log\left(2\pi\right))\left(s\right) &= \log\left(2\pi\right). \end{aligned}$$

Putting all this together, we have that

$$\begin{aligned} -s\mathcal{M}\left(x - \sum_{\rho}\frac{x^{\rho}}{\rho} - \log(2\pi)\right)(s) &= \frac{s}{1-s} + \sum_{\rho}\frac{s}{\rho\left(s-\rho\right)} + \log\left(2\pi\right) \\ &= -s(\mathcal{M}\psi)(s). \end{aligned}$$

We infer that

$$\mathcal{M}\left(x - \sum_{\rho}\frac{x^{\rho}}{\rho} - \log\left(2\pi\right)\right) = \mathcal{M}\psi.$$

Now we exploit the fact that the Mellin transform is injective and conclude that

$$x - \sum_{\rho}\frac{x^{\rho}}{\rho} - \log(2\pi) = \psi(x),$$

which is the required result. This completes the proof of the von Mangoldt explicit formula.

*The von Mangoldt explicit formula comes from the inverse Mellin transform of the logarithmic derivative of the Hadamard product for the zeta function.*

## 4.5. The logarithmic integral Li$(z)$

As preparation for the Riemann explicit formula, we need a careful discussion of the logarithmic integral Li$(z)$ of a complex variable. It is, of course, astonishing that Riemann used the function Li$(z)$ in the very early days of complex analysis. Riemann does not define Li$(z)$ in his article of 1859, so it was left to later researchers to figure out the definition. We start with the *exponential integral* as in Lebedev [**Leb**, §3.1].

**Definition 4.4.** Define the exponential integral as follows:

$$\operatorname{Ei}(z) := \int_{-\infty}^{z} \frac{e^w}{w} dw, \qquad |\arg(-z)| < \pi,$$

where the integration is along any path $L$ in the $w$-plane with a cut along the positive real axis.

The function $\operatorname{Ei}(z)$ is holomorphic in the plane with a cut along the positive real axis. A possible choice of the path of integration is the infinite line segment

$$-\infty < \Re(w) \leq \Re(z), \qquad \Im(w) = \Im(z),$$

passing through the point $z$ and parallel to the real axis.

According to [**Leb**, p. 32], the values of $\operatorname{Ei}(z)$ on the upper and lower edges of the cut are, respectively,

$$\begin{aligned} \operatorname{Ei}(u+i0) &= \operatorname{Ei}_1(u) - i\pi, \qquad & u > 0, \\ \operatorname{Ei}(u-i0) &= \operatorname{Ei}_1(u) + i\pi, \qquad & u > 0, \end{aligned}$$

where $\operatorname{Ei}_1(u)$ is the real function defined by

$$\operatorname{Ei}_1(u) = \gamma + \log u + \sum_{n=1}^{\infty} \frac{u^n}{n!n}, \qquad u > 0.$$

This is in preparation for the next definition.

**Definition 4.5.** Let $s = u + it$. Define the logarithmic integral as follows:

$$\begin{aligned} \operatorname{Li}(x^s) &= \operatorname{Ei}(s \log x) + i\pi \quad \text{if} \quad \Im(s) > 0 \\ &= \operatorname{Ei}(s \log x) - i\pi \quad \text{if} \quad \Im(s) < 0. \end{aligned}$$

On the upper and lower edges of the cut we have

$$\begin{aligned} \operatorname{Li}(x^{u+i0}) &= \operatorname{Ei}((u+i0)\log x) + i\pi \\ &= \operatorname{Ei}_1(u \log x), \\ \operatorname{Li}(x^{u-i0}) &= \operatorname{Ei}((u-i0)\log x) - i\pi \\ &= \operatorname{Ei}_1(u \log x), \end{aligned}$$

which effects a smooth transition across the cut for the function $s \mapsto \operatorname{Li}(x^s)$.

Note that Definition 4.5 is *not* the definition of Lebedev [**Leb**, §3.4]. However, it appears to be the definition favored by Riemann: on this point, see p. 30 of the book by Edwards [**Ed**] and the note on p. 186 of [**R**]. Definition 4.5 is also the definition favored by the Mathematica programmers.

Let $x > 2$. We now consider the curve

$$t \longmapsto \mathrm{Li}(x^{1/2+it})$$

whose domain is the critical line $\Re(s) = 1/2$. Definition 4.5 allows us to analyse this curve. It will turn out to be a *double spiral*.

**Theorem 4.6.** *The map* $t \longmapsto \mathrm{Li}\left(x^{1/2+it}\right)$ *defines a double spiral for which*

$$\lim_{t\to\infty} \mathrm{Li}\left(x^{1/2+it}\right) = i\pi \quad \textit{and} \quad \lim_{t\to\infty} \mathrm{Li}\left(x^{1/2-it}\right) = -i\pi.$$

*The upper arm spirals counterclockwise to the limit point* $i\pi$ *and the lower arm spirals clockwise to the limit point* $-i\pi$*. The upper arm is the mirror image, with respect to the real axis given by* $\Im(s) = 0$*, of the lower arm. The double spiral meets the real axis at the point*

$$\mathrm{Ei}_1((\log x)/2) = \gamma - \log 2 + \log\log x + \sum_{n+1}^{\infty} \frac{(\log x)^n}{2^n n! n}.$$

**Proof.** Fix $t \in \mathbb{R}$. Let $s = (u + it)\log x$. We have

$$\frac{ds}{s} = \frac{du}{u+it}$$

and $-\infty < u \le 1/2$. We have

$$\mathrm{Ei}((1/2+it)\log x) = \int_{-\infty}^{(\frac{1}{2}+it)\log x} \frac{e^s}{s} ds \tag{4.7}$$

$$= \int_{-\infty}^{\frac{1}{2}} \frac{x^u x^{it}}{u+it} du \tag{4.8}$$

$$= x^{it} \int_{-\infty}^{\frac{1}{2}} \frac{x^u}{u+it} du \tag{4.9}$$

and

$$\begin{aligned}|\mathrm{Ei}((u+it)\log x)| &= \left| x^{it}\int_{-\infty}^{1/2}\frac{x^u}{u+it}du\right| \\ &\leq \frac{1}{t}\int_{-\infty}^{1/2}x^u du \\ &= \frac{1}{t}\left(\frac{x^{1/2}}{\log x}\right).\end{aligned}$$

Then $\mathrm{Li}(x^{1/2+it}) \longrightarrow i\pi$ as $t \longrightarrow \infty$. Now suppose that $s = (v-it)\log x$. Then again we have

$$\begin{aligned}\mathrm{Ei}((1/2-it)\log x) &= \int_{-\infty}^{\frac{1}{2}}\frac{x^v x^{-it}}{v-it}dv \\ &= \overline{\mathrm{Ei}((1/2+it)\log x)}\end{aligned}$$

and

$$\begin{aligned}|\mathrm{Ei}((1/2-it)\log x)| &= \left| x^{-it}\int_{-\infty}^{1/2}\frac{x^v}{v-it}dv\right| \\ &\leq \frac{1}{t}\int_{-\infty}^{1/2}x^u du \\ &= \frac{1}{t}\left(\frac{x^{1/2}}{\log x}\right).\end{aligned}$$

Therefore we have $\mathrm{Li}(x^{1/2-it}) \longrightarrow -i\pi$ as $t \longrightarrow \infty$.

The phase of the complex number $\mathrm{Li}(x^{1/2+it})$ is, in view of (4.7), dominated by the phase of $x^{it}$, which is $t\log x$. This phase steadily increases as $t \longrightarrow \infty$ and indicates that we do indeed have a spiral curve. □

The zeta zeros which lie on the critical line can now be plotted on the double spiral. For a vivid graphical display, go to `www.demonstrations.wolfram.com` and enter *Logarithmic Integral on the critical line* in the search tab. If you download the *Wolfram Player* app, then you can vary the parameter $x$ and the range of $t$. You can also zoom towards the limit point $i\pi$. Strongly recommended!

Consider the complex conjugate zeta zeros $\rho, \overline{\rho}$. Let $\rho = \beta + i\gamma$. We have

$$\begin{aligned}\mathrm{Ei}((\beta - it)\log x) &= \int_{-\infty}^{\beta} \frac{x^v x^{-it}}{v - it} dv \\ &= \overline{\mathrm{Ei}((\beta + it)\log x)}\end{aligned}$$

and then

$$\mathrm{Li}(x^{\overline{\rho}}) = \overline{\mathrm{Li}(x^{\rho})}$$

which is compatible with the reflection principle. We also have

$$\mathrm{Li}(x^{\overline{\rho}}) + \mathrm{Li}(x^{\rho}) = 2\Re(\mathrm{Li}(x^{\rho})).$$

It is these real numbers, one for each complex zeta zero $\rho$, which make their contributions to the right-hand side of the Riemann formula, as we will see in the next section.

## 4.6. The Riemann formula

**Definition 4.7.** Let $x > 2$. We define

$$J(x) := \sum_{k \geq 0} \frac{\pi\left(x^{1/k}\right)}{k}$$

where $\pi(x)$ denotes the number of primes less than or equal to $x$.

This is a *finite series*: the series terminates as soon as $x^{1/k} < 2$.

**Theorem 4.8.** *The Riemann explicit formula is given by*

$$J(x) = \mathrm{Li}(x) - \sum_{\rho} \mathrm{Li}(x^{\rho}) - \log 2 + \int_{x}^{\infty} \frac{dt}{t(t^2 - 1)\log t}$$

*where the summation is over the complex zeta zeros $\rho$.*

Before moving on to a proof this formula, we attempt to shed some light on the individual terms, sometimes stating the obvious.

- The first term $\mathrm{Li}(x)$ is monotonic increasing.
- The second term is the most interesting. When we pair the terms coming from complex conjugate zeta zeros, we obtain the conditionally convergent series

$$-2\sum \Re(\mathrm{Li}(x^{\rho}))$$

summed over the zeros in the critical strip. Is the sum of this series positive or negative? Does it dominate, or is it dominated by, the first term $\mathrm{Li}(x)$? We will address this issue shortly. It is the key to understanding this formula.

- The third term is the constant $-\log 2$.
- The fourth term is

$$f(x) := \int_x^\infty \frac{dt}{t\,(t^2-1)\log t}.$$

This term $f(x)$ is monotonic decreasing, with $f(2) = 0.14001$ and

$$\lim_{x\to\infty} f(x) = 0.$$

The fourth term has an interesting interpretation. If we use the substitution $u = \log t$, then we can write

$$\begin{aligned}\int_x^\infty \frac{dt}{t\,(t^2-1)\log t} &= \int_{\log x}^\infty \frac{du}{u\,(e^{2u}-1)}\\ &= \int_{\log x}^\infty \frac{e^{2u}}{1-e^{-2u}}\frac{du}{u}\\ &= \int_{\log x}^\infty \sum_{n=1}^\infty \frac{e^{-2nu}du}{u}\\ &= \sum_{n=1}^\infty \int_{\log x}^\infty \frac{e^{-2nu}du}{u}.\end{aligned}$$

With a second substitution $t = e^{-2nu}$ we obtain

$$\begin{aligned}\sum_{n=1}^\infty \int_{\log x}^\infty \frac{e^{-2nu}du}{u} &= \sum_{n=1}^\infty \int_{x^{-2n}}^0 \frac{dt}{\log t}\\ &= -\sum_{n=1}^\infty \int_0^{x^{-2n}} \frac{dt}{\log t}\\ &= -\sum_{n=1}^\infty \mathrm{Li}\left(x^{-2n}\right),\end{aligned}$$

which is the second term in the Riemann formula, but now the summation is over the *trivial zeta zeros*. So we can, if we choose, write the Riemann explicit formula in the following concise way:

$$J(x) = \mathrm{Li}(x) - \sum_{\rho} \mathrm{Li}(x^{\rho}) - \log 2$$

where the sum is over all the zeta zeros.

We now have to extricate $\pi(x)$ from the finite series $J(x)$. The method is *Möbius inversion*.

**Definition 4.9.** We define the Möbius function $\mu$ as follows:

(1) $\mu(1) = 1$.
(2) $\mu(n) = 0$ if $n$ has a squared factor.
(3) Given distinct primes $p_1, ..., p_k$, then

$$\mu(p_1 \cdot p_2 \cdots p_k) = (-1)^k.$$

**Lemma 4.10.** *Let $\mu$ denote the Möbius function. Given a positive integer $n$, we have*

$$\sum_{d|n} \mu(d) = 0$$

*where $d|n$ means that $d$ divides $n$.*

**Proof.** Suppose that $n > 1$ and that $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where $p_1, ..., p_k$ are distinct prime numbers. Then we have that

$$\begin{aligned}\sum_{d|n} \mu(d) &= 1 + \sum_{i \geq 1} \mu(p_i) + \sum_{i \geq 1} \sum_{j \geq 1} \mu(p_i p_j) + \cdots + \mu\left(\prod_{i=1}^{k} p_i\right) \\ &= 1 - k + \binom{k}{2} + \cdots + (-1)^k \\ &= (1-1)^k \\ &= 0.\end{aligned}$$

□

We immediately exploit this result. Let $k$ be determined by the conditions $x, x^{1/2}, \dots, x^{1/k} > 2$ but $x^{1/(k+1)} < 2$. Then we have

$$\begin{aligned}\sum_{j=1}^{k} \frac{\mu(j)}{j} J\left(x^{1/j}\right) &= \sum_{j=1}^{k} \frac{\mu(j)}{j} \sum_{i=1}^{k} \frac{\pi\left(x^{1/ij}\right)}{i} \\ &= \sum_{n=1}^{k^2} \sum_{d|n} \frac{\mu(d)}{n} \pi\left(x^{1/n}\right) \\ &= \pi(x)\end{aligned}$$

with the change of variables $n = ij$. This expression for $\pi(x)$ is a finite series in the terms $J(x), \dots, J(x^{1/k})$. Combining this with the Riemann formula, we obtain

$$\begin{aligned}\pi(x) &= \sum_{n=1}^{k} \frac{\mu(n)}{n} J\left(x^{1/n}\right), \\ J(x) &= \operatorname{Li}(x) - \sum_{\rho} \operatorname{Li}(x^{\rho}) - \log 2 + \int_{x}^{\infty} \frac{dt}{t(t^2-1)\log t}\end{aligned}$$

where the sum is over the complex zeta zeros.

This is one of the great formulas of all time, expressing as it does the number of primes up to $x$ in terms of the zeta zeros.

We recall the classical theorem:

$$\begin{aligned}0 &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \\ &= 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{10} - \frac{1}{11} - \frac{1}{13} + \frac{1}{14} + \frac{1}{15} - \cdots .\end{aligned}$$

This result is stated by Euler in his book [**E**, Chapter XV, p. 235] and was subsequently proved by von Mangoldt; on this point, see the discussion by Edwards [**Ed**, p. 92].

Thanks to this result, the constant $-\log 2$ will disappear after Möbius inversion. It is customary to ignore the fourth term in the Riemann formula, the very small term which tends to 0. In that case, we have

$$\pi(x) = \sum_{n=1}^{k} \frac{\mu(n)}{n} \left[ \mathrm{Li}(x^{1/n}) - 2 \sum_{\rho} \Re(\mathrm{Li}(x^{\rho/n})) \right] \tag{4.10}$$

where the summation is over the zeta zeros in the upper half-plane.

We shall look carefully at this extraordinary formula for large values of $x$. It certainly looks as if $\mathrm{Li}(x)$ is going to win; i.e., it looks as if $\pi(x) < \mathrm{Li}(x)$ for all $x$. For $\mathrm{Li}(x)$ is immediately followed by three large negative terms corresponding to the primes $n = 2, 3, 5$.

The only way that $\pi(x)$ is going to beat $\mathrm{Li}(x)$ is if the zeta zeros arrange themselves on finitely many double spirals $\mathcal{S}_1, \ldots, \mathcal{S}_k$ in a very particular way. According to Littlewood's theorem, this is precisely what can happen! From a naive point of view, one would expect a great deal of cancellation, and indeed most of the time, this kind of cancellation does take place. But there are certain intervals on the real line for which the real values on these spirals *reinforce each other* sufficiently to dominate $\mathrm{Li}(x)$. This reinforcement must be *negative* in order to dominate the positive term $\mathrm{Li}(x)$.

Such intervals occur only when $x$ is very large, as we will see in the next sections.

Choose and fix $x > 2$ and then think of $x$ as large. To be clear about the finitely many double spirals, first note that $x^{\rho/j} = (x^{1/j})^{\rho}$. Now consider the double spiral

$$\mathcal{S} : t \longmapsto \mathrm{Li}\left(y^{\frac{1}{2}+it}\right)$$

and allow $y$ to take the values $x, x^{1/2}, x^{1/3}, \ldots, x^{1/k}$. This will create double spirals $\mathcal{S}_1, \ldots, \mathcal{S}_k$.

These double spirals are getting smaller, for the real numbers

$$\mathcal{S}_n \cap \text{real axis}$$

decrease as $n$ increases; see Theorem (4.6). To be precise, we have

$$\mathcal{S}_n \cap \text{real axis} = \mathrm{Ei}_1((\log x^{1/n})/2) = \mathrm{Ei}_1((\log x)/2n).$$

We also have to *scale* each double spiral $\mathcal{S}_j$ by $\frac{\mu(j)}{j}$. The complex numbers $\mathrm{Li}(x^{\rho/n})$ will lie on the double spiral $\mathcal{S}_n$ whenever $\rho$ is a zeta zero on the critical line.

Here is a thought experiment. Imagine looking at a large number of these double spirals $\mathcal{S}_1, \ldots, \mathcal{S}_k$. Assume the Riemann hypothesis, and plot the complex zeta zeros on all of these spirals. If you look at this configuration from a great distance, then these millions of points will usually arrange themselves equally on each side of the imaginary axis $\Im(s) = 0$. However, for certain values of $x$, these millions of points will have a tendency to arrange themselves to the left of the imaginary axis. This phenomenon, coupled with the negative sign attached to the oscillating terms in the Riemann formula, will be enough to offset the impact of the first three negative terms

$$-\frac{1}{2} \cdot \mathrm{Li}\left(x^{1/2}\right) - \frac{1}{3} \cdot \mathrm{Li}\left(x^{1/3}\right) - \frac{1}{5} \cdot \mathrm{Li}\left(x^{1/5}\right)$$

in the Riemann formula, thereby resulting in $\pi(x) > \mathrm{Li}(x)$; i.e., the prime number theorem undercounts the number of primes. This behavior of the complex zeta zeros is remarkably subtle.

## 4.7. Origin of the Riemann explicit formula

We start with the Euler product. We have

$$\begin{aligned}
\log \zeta(s) &= -\sum_p \log\left(1 - p^{-s}\right) \\
&= \sum_p \sum_{k \geq 1} \frac{1}{k} p^{-ks} \\
&= \int_0^\infty x^{-s} dJ(x) \\
&= s \int_0^\infty J(x)\, x^{-s} \frac{dx}{x} \\
&= s\, (\mathcal{M}J)(s)
\end{aligned}$$

so that

$$(\mathcal{M}J)(s) = \frac{\log \zeta(s)}{s}$$

where $\mathcal{M}$ is the Mellin transform.[1]

---

[1] At the time when Riemann was using the Mellin transform and its inverse, the Finnish mathematician Mellin was five years old! Clearly a precocious child.

We apply the inverse Mellin transform with $c > 1$:

$$J(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\log \zeta(s)}{s} x^s ds. \tag{4.11}$$

If

$$s = \frac{1}{2} + iz,$$

then

$$\xi(z) := \frac{1}{2} s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s) \tag{4.12}$$

is an entire function of $z$. Its zeros are given by

$$s = \rho, \qquad z = -i(\rho - 1/2) = \tau.$$

They lie symmetrically about the origin and are real if the Riemann hypothesis is true; and

$$\xi(z) = \xi(0) \prod_{\tau} (1 - z^2/\tau^2). \tag{4.13}$$

From (4.12) and (4.13) we obtain

$$\begin{aligned} \log \zeta(s) = &- \log(s-1) - \log \Gamma(1 + s/2) + \frac{1}{2} s \log \pi + \log \xi(0) \\ &+ \sum \log(1 + (s - 1/2)^2/\tau^2) \end{aligned} \tag{4.14}$$

and Riemann substitutes from (4.14) into (4.11) and evaluates the terms of the resulting series separately. The dominant term of the Riemann formula results from the term $-\log(s-1)$; see [**H**, §2.12]. For the explicit evaluations, see Edwards [**Ed**, §3.7].

It is interesting to compare the origin of the von Mangoldt formula with the origin of the Riemann formula. We recall:

*the von Mangoldt explicit formula comes from the inverse Mellin transform of the logarithmic derivative of the Hadamard product for the zeta function.*

For comparison, *the Riemann explicit formula for $J(x)$ comes from the inverse Mellin transform of $s^{-1}$ times the logarithm of the Hadamard product for $\zeta(s)$. This naturally requires term-by-term integration. The detailed computations are in* [**Ed**, §3.7].

# Chapter 5

# The prime number theorem

## 5.1. The Riemann-Ramanujan approximation

Define

$$\begin{aligned} R(x) &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \cdot \mathrm{Li}(x^{1/n}) \\ &= \mathrm{Li}(x) - \frac{1}{2} \cdot \mathrm{Li}(x^{1/2}) - \frac{1}{3} \cdot \mathrm{Li}(x^{1/3}) - \frac{1}{5} \cdot \mathrm{Li}(x^{1/5}) \\ & \quad + \frac{1}{6} \cdot \mathrm{Li}(x(^{1/6}) - \frac{1}{7} \cdot \mathrm{Li}(x^{1/7}) + \frac{1}{10} \cdot \mathrm{Li}(x^{1/10}) \\ & \quad - \frac{1}{11} \cdot \mathrm{Li}(x^{1/11}) - \frac{1}{13} \cdot \mathrm{Li}(x^{1/13}) + \cdots . \end{aligned}$$

If, in the Riemann explicit formula, you ignore all the oscillating terms, then you obtain the *approximation*

$$\pi(x) \approx R(x)$$

which Riemann, in his memoir [**R**], compares favorably with the (Gaussian) approximation

$$\pi(x) \approx \mathrm{Li}(x).$$

Ramanujan's assertion [**H**, p. 26] is that

$$\pi(x) - R(x) = O(1),$$

i.e., that $\pi(x) - R(x)$ is bounded.

According to Hardy [**H**, p. 26], this formula is false. Hardy is full of admiration for Ramanujan's work; nevertheless, this was a famous occasion when Ramanujan's intuition let him down.

Overall, $R(x)$ is a good approximation to $\pi(x)$. However, it does not track the jumps of $\pi(x)$. Furthermore, when a gap occurs in the primes (as, for example, between 23 and 29), $R(x)$ keeps increasing even though $\pi(x)$ remains constant between primes.

If we add to $R(x)$ the first few oscillatory terms in the Riemann formula (4.10), then we get a new function that very closely matches the jumps and irregularities of $\pi(x)$. When $x$ is prime, this new function increases by about 1 near $x$, so, in effect, it "knows" where the primes are. And when a gap occurs, this new function tends to level out, again emulating the behavior of $\pi(x)$. This phenomenon, whereby the Riemann formula hugs the function $\pi(x)$, has to be seen to be believed.

To see this for yourself, go to `www.demonstrations.wolfram.com` and enter *How the Zeros of the Zeta Function Predict the Distribution of the Primes* in the search tab. If you download the *Wolfram Player* app, then you can vary the parameters. Even with the first 100 pairs of zeta zeros, this phenomenon is clearly visible. Try it for yourself!

## 5.2. Proof of the prime number theorem

The prime number theorem is an *asymptotic* statement. Recall that $f(x) \sim g(x)$ as $x \to \infty$ means that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

**Theorem 5.1** (Prime number theorem). *We have*

$$\pi(x) \sim \operatorname{Li}(x) \quad \textit{as} \quad x \to \infty.$$

This theorem was proved (independently) by Hadamard and de la Vallée Poussin in 1896.

We will give Newman's proof of the prime number theorem. This proof is short and efficient. We will base our exposition on that of Zagier [**Zag**]. Zagier's exposition is concise and elegant: here we will include more details. In order to make this section self-contained, we will review and recall certain results from earlier sections.

We present the argument is a series of steps. Specifically, we prove a sequence of properties of the three functions

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Phi(s) := \sum_p \frac{\log p}{p^s}, \quad \vartheta(x) := \sum_{p \le x} \log p$$

with $s \in \mathbb{C}, x \in \mathbb{R}$.

**Lemma 5.2.** *We have*

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}$$

*for* $\Re(s) > 1$*.*

**Proof.** We already proved this; here is a précis of the proof. From unique factorization and the absolute convergence of $\zeta(s)$ we have

$$\begin{aligned} \zeta(s) &= \sum_{r_2, r_3, \ldots \ge 0} (2^{r_2} 3^{r_3} \cdots)^{-s} \\ &= \prod_p \left( \sum_{r \ge 0} p^{-rs} \right) \\ &= \prod_p \frac{1}{1 - p^{-s}} \end{aligned}$$

for $\Re(s) > 1$. □

**Lemma 5.3.** *The function*

$$\zeta(s) - \frac{1}{s-1}$$

*extends holomorphically to* $\Re(s) > 0$*.*

**Proof.** We already proved this too; here is a précis of the proof. For $\Re(s) > 1$ we have

$$\begin{aligned}\zeta(s) - \frac{1}{s-1} &= \sum_{n=1}^{\infty} \frac{1}{n^s} - \int_1^{\infty} \frac{1}{x^s} dx \\ &= \sum_{n=1}^{\infty} \int_n^{n+1} \left( \frac{1}{n^s} - \frac{1}{x^s} \right) dx.\end{aligned}$$

The series in the last line converges absolutely for $\Re(s) > 0$ because

$$\begin{aligned}\left| \int_n^{n+1} \left( \frac{1}{n^s} - \frac{1}{x^s} \right) dx \right| &= \left| s \int_n^{n+1} \int_n^x \frac{du}{u^{s+1}} dx \right| \\ &\leq \max_{n \leq u \leq n+1} \left| \frac{s}{u^{s+1}} \right| \\ &= \frac{|s|}{n^{\Re(s)+1}}\end{aligned}$$

by the mean value theorem. ☐

**Lemma 5.4.** *We have* $\vartheta(x) < (\log 4)x$.

**Proof.** It is enough to consider the natural numbers. For $n \in \mathbb{N}$ we have

$$\begin{aligned}2^{2m+1} &= (1+1)^{2m+1} \\ &= 1 + \cdots + \binom{2m+1}{m} + \binom{2m+1}{m} + \cdots + 1.\end{aligned}$$

Let $M = \binom{2m+1}{m}$ be the central binomial coefficient. Then we have

$$2M < 2^{2m+1} \implies M < 2^{2m} \implies \log M < 2m \log 2.$$

Now consider

$$\vartheta(2m+1) - \vartheta(m+1) = \sum_{m+1 < p \leq 2m+1} \log p.$$

Each such prime occurs in the numerator but not the denominator of

$$M = \binom{2m+1}{m}$$

so the product of such primes divides $M$:

$$\prod_{m+1<p\le 2m+1} p \,|\, M$$

so we have

$$\sum_{m+1<p\le 2m+1} \log p < \log M$$

and therefore we have

$$\vartheta(2m+1) - \vartheta(m+1) < 2m\log 2.$$

Now use induction, first taking $n = 2m+1$, an odd number:

$$\begin{aligned} \vartheta(n) &= \vartheta(2m+1) - \vartheta(m+1) + \vartheta(m+1) \\ &< 2m\log 2 + (m+1)\log 4 \\ &= m\log 4 + (m+1)\log 4 \\ &= n\log 4. \end{aligned}$$

If $n$ is even, then we have

$$\begin{aligned} \vartheta(n) &= \vartheta(n-1) \\ &< (n-1)\log 4 \\ &< n\log 4. \end{aligned}$$

□

**Definition 5.5.** The functions $f$ and $g$ are as follows:

$$\begin{aligned} f(t) : &= \frac{\vartheta(e^t)}{e^t} - 1 && (t \in \mathbb{R}), \\ g(s-1) : &= \frac{\Phi(s)}{s} - \frac{1}{s-1} && \text{for} \quad \Re(s) > 1. \end{aligned}$$

**Lemma 5.6.** *We have that $g(s-1)$ is holomorphic for $\Re(s) > 1$ and $g(0)$ is finite.*

**Proof.** For $\Re(s) > 1$, the convergent product in Lemma 5.2 implies that $\zeta(s) \neq 0$ and that

$$\begin{aligned} -\frac{\zeta'(s)}{\zeta(s)} &= \sum_p \frac{\log p}{p^s - 1} \\ &= \Phi(s) + \sum_p \frac{\log p}{p^s(p^s-1)}. \end{aligned}$$

The final sum converges for $\Re(s) > 1/2$, so this and Lemma 5.3 imply that $\Phi(s)$ extends meromorphically to $\Re(s) > 1/2$, with poles only at $s = 1$ and at the zeros of $\zeta(s)$. However, $\zeta$ is free of zeros on $\Re(s) = 1$, by Theorem 2.8. So the only possible pole of $\Phi(s)$ is at $s = 1$.

By the elementary Exercise 1.7, the left-hand side of the above equation has a simple pole at $s = 1$ with residue 1. It follows that $\Phi(s) - 1/(s-1)$ has a removable singularity at $s = 1$. The same is true for

$$g(s-1) = \frac{\Phi(s)}{s} - \frac{1}{s-1}$$

so that $g(s-1)$ is holomorphic for $\Re(s) > 1$ and $g(0)$ is finite. □

**Definition 5.7.** The Laplace transform is defined as

$$(\mathcal{L}F)(s) = \int_0^\infty e^{-st} F(t) dt$$

under certain conditions on the function $F$ and the variable $s$.

**Lemma 5.8.** *We have*

$$(\mathcal{L}f)(s-1) = g(s-1) \quad \textit{for} \quad \Re(s) > 1.$$

**Proof.** For $\Re(s) > 1$ we have

$$\begin{aligned} \Phi(s) &= \sum_p \frac{\log p}{p^s} \\ &= \int_1^\infty \frac{d\vartheta(x)}{x^s} \\ &= s \int_1^\infty \frac{\vartheta(x)}{x^{s+1}} dx \\ &= s \int_0^\infty e^{-st} \vartheta(e^t) dt \end{aligned}$$

and

$$\begin{aligned} (\mathcal{L}f)(s-1) &= \int_0^\infty e^{-(s-1)t} f(t) dt \\ &= \int_0^\infty e^{-st} \vartheta(e^t) dt - \int_0^\infty e^{-(s-1)t} dt \\ &= \frac{\Phi(s)}{s} - \frac{1}{s-1}. \end{aligned}$$

□

**Lemma 5.9.** *We have* $(\mathcal{L}f)(0)$ *is finite and equals* $g(0)$. *In other words, the basic Laplace transform equation in Lemma* 5.8 *remains true when* $s = 1$.

**Proof.** For $T > 0$ set

$$g_T(z) = \int_0^T f(t)e^{-zt}dt.$$

This is clearly holomorphic for all $z$. We must show that

$$\lim_{T\to\infty} g_T(0) = g(0).$$

Let $R$ be large and let $C$ be the boundary of the region

$$\{z \in \mathbb{C} : |z| \le R, \Re(z) \ge -\delta\}$$

where $\delta > 0$ is small enough (depending on $R$) so that $g(z)$ is holomorphic in and on $C$. Then

$$g(0) - g_T(0) = \frac{1}{2\pi i}\int_C (g(z) - g_T(z))e^{zT}(1 + z^2/R^2)\frac{dz}{z}$$

by Cauchy's theorem. On the semicircle $C_+ = C \cap \{\Re(z) > 0\}$ the integral is bounded by $2B/R^2$, where $B = \max_{t\ge 0}|f(t)|$, because

$$\begin{aligned} |g(z) - g_T(z)| &= \left|\int_T^\infty f(t)e^{-zt}dt\right| \\ &\le B\int_T^\infty |e^{-zt}|dt \\ &= \frac{Be^{-\Re(z)T}}{\Re(z)} \qquad (\Re(z) > 0) \end{aligned}$$

and

$$\left|e^{zT}\left(1 + \frac{z^2}{R^2}\right)\frac{1}{z}\right| = e^{\Re(z)T}\cdot\frac{2\Re(z)}{R^2}.$$

Hence the contribution to $g(0) - g_T(0)$ from the integral over $C_+$ is bounded in absolute value by $B/R$. For the integral over $C_- = C \cap \{\Re(z) < 0\}$ we look at $g(z)$ and $g_T(z)$ separately. Since $g_T$ is entire, the path of integration for the integral involving $g_T$ can be replaced by the semicircle $C'_- = \{z \in \mathbb{C} : |z| = R, \Re(z) < 0\}$, and the integral over $C'_-$ is then bounded in absolute value by $2\pi B/R$ by exactly the same estimate as before since

$$\begin{aligned} |g_T(z)| &= \left|\int_0^T f(t)e^{-zt}dt\right| \\ &\leq B\int_{-\infty}^T |e^{-zt}|dt \\ &= \frac{Be^{-\Re(z)T}}{|\Re(z)|} \qquad (\Re(z) < 0). \end{aligned}$$

Finally, the remaining integral over $C_-$ tends to 0 as $T \to \infty$ because the integrand is the product of the function $g(z)(1+z^2/R^2)/z$, which is independent of $T$, and the function $e^{zT}$, which goes to 0 rapidly and uniformly on compact sets as $T \to \infty$ in the half-plane $\Re(z) < 0$. Hence

$$\limsup_{T\to\infty} |g(0) - g_T(0)| \leq 2B/R.$$

Since $R$ is arbitrary, this proves the theorem. □

**Lemma 5.10.** *The following is a convergent integral:*

$$\int_1^\infty \frac{\vartheta(x) - x}{x^2}.$$

**Proof.** We note that the function $f$ is bounded by 1, for

$$f(t) = \frac{\vartheta(e^t)}{e^t} - 1 < \log 4 - 1 < 1$$

by Lemma 5.4. Furthermore, we have, by Lemma 5.9

$$\begin{aligned} g(0) &= (\mathcal{L}f)(0) \\ &= \int_0^\infty f(t)dt \\ &= \int_0^\infty \frac{\vartheta(e^t) - e^t}{e^t}dt \\ &= \int_1^\infty \frac{\vartheta(x) - x}{x^2}dx \qquad (x = e^t). \end{aligned}$$ □

**Lemma 5.11.** *We have*

$$\vartheta(x) \sim x.$$

**Proof.** Assume that for some $\lambda > 1$ there are arbitrarily large $x$ with $\vartheta(x) \geq \lambda x$. Since $\vartheta$ is nondecreasing, we have

$$\begin{aligned} \int_x^{\lambda x} \frac{\vartheta(t) - t}{t^2} dt &\geq \int_x^{\lambda x} \frac{\lambda x - t}{t^2} dt \\ &= \int_1^{\lambda} \frac{\lambda - t}{t^2} dt \\ &> 0 \end{aligned}$$

for such $x$, contradicting lemma 5.10. Similarly, the inequality $\vartheta(x) \leq \lambda x$ with $\lambda < 1$ would imply

$$\begin{aligned} \int_{\lambda x}^{x} \frac{\vartheta(t) - t}{t^2} dt &\leq \int_{\lambda x}^{x} \frac{\lambda x - t}{t^2} dt \\ &= \int_{\lambda}^{1} \frac{\lambda - t}{t^2} dt \\ &< 0, \end{aligned}$$

again a contradiction for $\lambda$ fixed and $x$ big enough. ☐

To finish the proof of the prime number theorem, we note that

$$\begin{aligned} \vartheta(x) &= \sum_{p \leq x} \log p \\ &\leq \sum_{p \leq x} \log x \\ &= \pi(x) \log x. \end{aligned}$$

Rearranging, we obtain

$$1 \leq \frac{\pi(x) \log x}{\vartheta(x)}. \tag{5.1}$$

Given $0 < \delta < 1$, we have

$$\begin{aligned} \vartheta(x) &\geq \sum_{x^{1-\delta} \leq p \leq x} \log p \\ &\geq \sum_{x^{1-\delta} \leq p \leq x} (1 - \delta) \log x \\ &= (1 - \delta)(\pi(x) - \pi(x^{1-\delta})) \log x \\ &\geq (1 - \delta)(\pi(x) - x^{1-\delta}) \log x. \end{aligned}$$

Rearranging, we obtain

$$\frac{\pi(x)\log x}{\vartheta(x)} \leq \frac{\log x}{\vartheta(x)} \cdot x^{1-\delta} + \frac{1}{1-\delta}. \tag{5.2}$$

We have

$$\frac{\log x}{\vartheta(x)} \cdot x^{1-\delta} = \frac{x}{\vartheta(x)} \cdot \frac{\log x}{x^{\delta}} \to 0 \qquad \text{as} \quad x \to \infty$$

by Lemma 5.11.

Combining (5.1) and (5.2), we get

$$1 \leq \frac{\pi(x)\log x}{\vartheta(x)} \leq \frac{\log x}{\vartheta(x)} \cdot x^{1-\delta} + \frac{1}{1-\delta}.$$

Given $\epsilon > 0$, choose $\delta > 0$ for which $1/(1-\delta) < 1 + \epsilon/2$. Then, there exists $x_0$ such that

$$\begin{aligned} x > x_0 \implies 1 &\leq \frac{\pi(x)\log x}{\vartheta(x)} \\ &< \frac{\epsilon}{2} + \left(1 + \frac{\epsilon}{2}\right) \\ &= 1 + \epsilon; \end{aligned}$$

i.e.,

$$\pi(x) \sim \frac{\vartheta(x)}{\log x} \sim \frac{x}{\log x}$$

again using Lemma 5.11. Finally, we have

$$\frac{x}{\log x} \sim \mathrm{Li}(x)$$

by the elementary Exercise 6.1. This concludes the proof of the prime number theorem.

If $y = \pi(x)$, then we have, successively,

$$\begin{aligned} y &\sim x/\log x, \\ \frac{y \log x}{x} &\to 1 \quad \text{as} \quad x \to \infty, \\ \log y + \log\log x - \log x &\to 0 \quad \text{as} \quad x \to \infty, \\ \frac{\log y}{\log x} + \frac{\log\log x}{\log x} &\to 1 \quad \text{as} \quad x \to \infty, \\ \frac{\log y}{\log x} &\to 1 \quad \text{as} \quad x \to \infty. \end{aligned}$$

But we have $(y \log x)/x \to 1$; therefore,

$$\frac{y \log y}{x} \to 1$$

as $x \to \infty$. Let $x = p_n$, the $n$th prime. Then $y = \pi(p_n) = n$ and so

$$\frac{n \log n}{p_n} \to 1$$

as $n \to \infty$. This establishes the following asymptotic statement:

$$p_n \sim n \log n.$$

We may therefore state that the average spacing between primes around $n$ is $\log n$. Now multiply $n$ by 10: the average spacing between primes around $10n$ is $\log 10n = \log n + \log 10$. Therefore:

**Theorem 5.12.** *If you increase a natural number by a factor of* 10*, then the average spacing between primes will increase by* $\log 10$*. This increase is independent of the chosen number.*

The prime numbers are "irregularly spaced", but Theorem 5.12 pinpoints a certain uniform behavior underlying this irregularity. This can be regarded as the down-to-earth interpretation of the prime number theorem. Note that $\log 10 \approx 2.3$. Upon multiplication by 10, the average spacing between primes will increase by 2.3.

## Chapter 6

# Oscillation of $\pi(x) - \mathrm{Li}(x)$

### 6.1. Littlewood's theorem

As preparation for the statement of Littlewood's theorem, we need some notation. Let $\pi(x)$ denote the number of primes less than or equal to $x$, and let $\mathrm{Li}(x)$ denote the logarithmic integral. The notation $f(x) = \Omega_{\pm} g(x)$ means that

$$\limsup_{x\to\infty} f(x)/g(x) > 0, \qquad \liminf_{x\to\infty} f(x)/g(x) < 0.$$

There was, in 1914, overwhelming numerical evidence that $\pi(x) < \mathrm{Li}(x)$ for all $x$. In spite of this, Littlewood [**Li**] announced that

$$\pi(x) - \mathrm{Li}(x) = \Omega_{\pm}(x^{1/2}(\log x)^{-1} \log\log\log x).$$

This implies that $\pi(x) - \mathrm{Li}(x)$ changes sign infinitely often.

From a qualitative point of view, the picture which emerges is as follows. The average spacing between primes around $x$ is $\log x$. We recall that we have defined the Skewes number $\Xi$ to be the first crossover. As we approach the first crossover $\Xi$, the primes will start to get a little more crowded together, and the average spacing will be a little less than $\log x$, sufficiently for $\pi(x)$ to exceed $\mathrm{Li}(x)$ at the first crossover. As we move away from the first crossover, the primes will gradually thin out, until $\pi(x)$ is less than $\mathrm{Li}(x)$. Then, very gradually, the primes will start to get a little more crowded together, sufficiently for $\pi(x)$ to exceed $\mathrm{Li}(x)$ at the second crossover. This remarkable process will repeat itself infinitely often.

In the context of the Riemann explicit formula, there will be a parallel phenomenon for the zeta zeros. The relevant contribution of the complex zeta zeros will be

$$-2\sum_{\rho}\Re\left(\mathrm{Li}(x^{\rho})\right).$$

The issue is one of *reinforcement*, as opposed to cancellation, among the oscillating terms. Around the first crossover $\Xi$, the terms in this infinite series will reinforce one another, in such a way that the sum of this infinite series will be sufficiently positive to dominate the three large negative terms

$$-\frac{1}{2}\mathrm{Li}(x^{1/2}) - \frac{1}{3}\mathrm{Li}(x^{1/3}) - \frac{1}{5}\mathrm{Li}(x^{1/5})$$

with which the Riemann formula begins. In that case, $\pi(x)$ will exceed $\mathrm{Li}(x)$. This parallel phenomenon for the zeta zeros surely deserves further research.

This chapter contains two oscillation theorems, the first due to Littlewood [**Li**], the second due to Lehman [**Leh**]. We will outline a proof of Littlewood's theorem and provide a refinement of Lehman's theorem.

In Lemma 6.2 below, we have set $\rho = 1/2 + i\gamma$ where $\rho$ is a complex zeta zero.

The method is to *average* the difference $\psi(u) - u$ over a chosen interval $[e^{-\delta}x, e^{\delta}x]$. If this value is positive, then there must be a point within the interval where $\psi(u)$ exceeds $u$.

**Definition 6.1.** This average is denoted as follows:

$$F(x,\delta) := \frac{1}{x(e^{\delta} - e^{-\delta})}\int_{e^{-\delta}x}^{e^{\delta}x}(\psi(u) - u)du.$$

The endpoints of the interval are chosen so that the resulting formulas work out nicely. The positive real number $\delta$ is fixed.

**Lemma 6.2.** *If the RH* (*Riemann hypothesis*) *is true, then we have*

$$F(x,\delta) = -2x^{1/2}\sum_{\gamma>0}\frac{\sin(\gamma\delta)}{\gamma\delta}\frac{\sin(\gamma\log x)}{\gamma} + O(x^{1/2})$$

*uniformly for* $x \geq 4, 1/(2x) \leq \delta \leq 1/2$.

**Proof.** For this lemma, and the theorem which follows, we will outline the proof. Further details may be found in §15.2 of the book by Montgomery and Vaughan [**MV**] and the online class notes of Titichetrakun [**TT**].

The starting point is the von Mangoldt explicit formula (4.1):

$$\psi(u) = u - \sum_{\rho}\frac{u^\rho}{\rho} - \frac{1}{2}\log\left(1-\frac{1}{u^2}\right) - \log 2\pi.$$

Integrating from 0 to $x$, we obtain

$$\int_0^x (\psi(u)-u)du = -\sum_{\rho}\frac{x^{\rho+1}}{\rho(\rho+1)} - (\log 2\pi)x + O(1)$$

for $x \geq 2$. We replace $x$ by $e^{\pm\delta}x$ and find the difference to obtain

$$\frac{e^{\delta(\rho+1)} - e^{-\delta(\rho+1)}}{2\delta\rho(\rho+1)}x^\rho + O(1).$$

We appeal to RH, and rewrite this term as

$$-x^{1/2}\left(\frac{\delta}{\sinh\delta}\right)\sum_{\rho}\frac{e^{\delta(\rho+1)} - e^{-\delta(\rho+1)}}{2\delta\rho(\rho+1)}x^{i\gamma} + O(1).$$

Elementary estimates, as in [**MV**, pp. 477–478], lead to the expression

$$-x^{1/2}\sum_{\rho}\frac{\sin\gamma\delta}{\gamma\delta}\cdot\frac{x^{i\gamma}}{i\gamma} + O(x^{1/2}).$$

By combining the contributions of $\gamma$ and $-\gamma$, we obtain

$$-2x^{1/2}\sum_{\gamma>0}\frac{\sin(\gamma\delta)}{\gamma\delta}\frac{\sin(\gamma\log x)}{\gamma} + O(x^{1/2})$$

as required. ☐

**Theorem 6.3** (Littlewood)**.** *As $x \to \infty$,*

$$\psi(x) - x = \Omega_{\pm}(x^{1/2} \log\log\log x), \tag{6.1}$$

*and*

$$\pi(x) - \mathrm{Li}(x) = \Omega_{\pm}(x^{1/2}(\log x)^{-1} \log\log\log x). \tag{6.2}$$

**Proof.** Let $N$ denote a large integer and let $\delta = 1/N$ so that

$$F(x, 1/N) = -2x^{1/2} \sum_{\gamma > 0} \frac{\sin(\gamma/N)}{\gamma/N} \frac{\sin(\gamma \log x)}{\gamma} + O(x^{1/2}).$$

Now the terms

$$\left\{ \frac{\sin(\gamma/N)}{\gamma/N} \frac{\sin(\gamma \log x)}{\gamma} : \gamma > 0 \right\}$$

in this infinite series oscillate in sign, sometimes positive, sometimes negative, so it is hard to estimate the sum of this infinite series.

What if we could find a *positive* replacement for each term? What if many of the terms had modulus close to 1? Then we would have a chance of estimating the sum of this series. A theorem of Dirichlet [**MV**, Lemma 15.10] allows us make such a replacement.

Let $N$ be a large integer and set $T = N \log N$. Consider those numbers $\gamma(\log N)/2\pi$ for which $0 < \gamma \le T$. Define

$$K := N(T) \asymp T \log T.$$

By Dirichlet's theorem, there exists an integer $1 \le n \le N^K$ such that

$$\left|\left| \frac{\gamma n}{2\pi} \log N \right|\right| < 1/N$$

for all $0 < \gamma \le T$. Take $x = N^n e^{\pm 1/N}, \delta = 1/N$ and use the inequality

$$|\sin 2\pi\alpha - \sin 2\pi\beta| \le 2\pi ||\alpha - \beta||$$

where $||x||$ denotes the distance from $x$ to the nearest integer. We obtain

$$\begin{aligned}|\sin\gamma\log x \mp \sin(\gamma/N)| &\le 2\pi\left|\left|\frac{\gamma\log x \mp \gamma/N}{2\pi}\right|\right|\\ &= 2\pi\left|\left|\frac{\gamma(n\log N \pm 1/N)\mp\gamma/N}{2\pi}\right|\right|\\ &= 2\pi\left|\left|\frac{\gamma\log Nn}{2\pi}\right|\right|\\ &\le \frac{2\pi}{N}\end{aligned}$$

for $0<\gamma\le T$. Since

$$\sum_\gamma\left|\frac{\sin(\gamma/N)}{\gamma/N}\cdot\frac{1}{\gamma}\right| \ll (\log N)^2$$

and $\sum_{\gamma>T}1/\gamma^2 \ll T^{-1}\log T \ll 1/N$, we infer that

$$F(x,1/N) = \mp x^{1/2}N^{-1}\sum_{\gamma>0}\left(\frac{\sin\gamma/N}{\gamma/N}\right)^2 + O(x^{1/2}).$$

The sum over $\gamma$ is $\asymp N\log N$, so we get

$$F(x,1/N) = \pm 2x^{1/2}\log N + O(x^{1/2}).$$

But $x \le N^{N^K}e^{1/N}$ and $N(T)\asymp T\log T \asymp N(\log N)^2$, so that

$$\log\log x \ll N(\log N)^3,$$

and hence

$$\log N \ge (1+o(1))\log\log\log x.$$

The left-hand side in Lemma 6.2 is simply the average of $\psi(u)-u$ over a neighborhood of $x$. Since $x \gg N$ and $N$ is arbitrarily large, we have (6.1).

As for (6.2), we note that if RH holds, then (6.1) and (6.2) are equivalent in view of Theorem 13.2 in [**MV**]. If RH is false, then Theorem 15.2 in [**MV**] gives a stronger result. □

According to [**Ing**, Theorem 34], we have the stronger results

$$\limsup_{x\to\infty}\frac{\psi(x)-x}{x^{1/2}\log\log\log x} \ge 1/2$$

and

$$\liminf_{x\to\infty} \frac{\psi(x) - x}{x^{1/2} \log\log\log x} \leq -1/2$$

with similar inequalities for $\pi(x) - \mathrm{Li}(x)$.

## 6.2. Lehman's theorem

Lehman [**Leh**] chose to work with the explicit formula for $\pi(x) - \mathrm{Li}(x)$ as originally conceived in Riemann's paper. The goal is to develop a computational formula for the difference, which depends, to sufficient accuracy, on only a finite number of zeta zeros. It is easy to see that for any positive function $K(y)$ and any interval $(a, b)$,

$$\int_a^b K(y)\{\pi(y) - \mathrm{Li}(y)\}\, dy > 0$$

implies that $\pi(y) - \mathrm{Li}(y)$ is positive within $(a, b)$. Lehman showed that with a suitable choice of $K(y)$, the computation of this integral only required the first 12,000 zeta zeros to yield a dramatically improved upper bound for the Skewes number. Since Lehman's publication, other mathematicians have used his formula with some refinement to yield the best upper bound that is currently known.

Littlewood's method provided, even in principle, no definite number $X$ before which $\pi(x) - \mathrm{Li}(x)$ changes sign.

In the course of the 20th century, successive numerical upper bounds were found by Skewes [**S0**] conditional on RH, Skewes [**S**] unconditional, Lehman [**Leh**], and te Riele [**tR**]. In the course of the 21st century we have the contributions of Bays-Hudson [**BH**], Chao-Plymen [**CP**], and Saouter-Demichel [**SD**].

For Littlewood's own account of the discovery of the Skewes numbers, see [**L2**, pp. 110–112].

The smallest value of $x$ with $\pi(x) \geq \mathrm{Li}(x)$ will be denoted $\Xi$. It is known [**But**] that

$$10^{19} < \Xi.$$

We now explain the main idea in [**Leh**]. Lehman's theorem is an *integrated version of the Riemann explicit formula.* His method was to integrate the function $u \mapsto \pi(e^u) - \mathrm{Li}(e^u)$ against a Gaussian kernel over a carefully chosen interval $[\omega - \eta, \omega + \eta]$. The definite integral so obtained is denoted $I(\omega, \eta)$. Let $\rho = 1/2 + i\gamma$ denote a zeta zero with $\gamma > 0$ and let

$$H(T, \omega) := -2\Re \sum_{0<\gamma\leq T} \frac{e^{i\gamma\omega}}{\rho} e^{-\gamma^2/2\alpha}.$$

The $\alpha$ in this formula is related to the kernel chosen. Lehman proved the equality

$$I(\omega, \eta) = -1 + H(T, \omega) + R$$

together with an explicit estimate $|R| \leq \epsilon$. This creates the inequality

$$I(\omega, \eta) \geq H(T, \omega) - (1 + \epsilon).$$

The problem now is to prove that

$$H(T, \omega) > 1 + \epsilon. \tag{6.3}$$

If (6.3) holds, then $I(\omega, \eta) > 0$ and so there exists $x \in [e^{\omega-\eta}, e^{\omega+\eta}]$ for which $\pi(x) > \mathrm{Li}(x)$. In order to establish (6.3), numerical values of the zeta zeros with $|\gamma| < T$ are required. Each term in $H(T, \omega)$ is a complex number determined by a zeta zero. It is necessary that the real parts of these complex numbers, which are spiralling towards 0, reinforce each other sufficiently for (6.3) to hold. The only known way of establishing this is by numerical computation. When $T$ is large, this requires a computer.

The function $H(T, \omega)$ is an initial part of the series

$$H(\omega) := -2\Re \sum_{0<\gamma} \frac{e^{i\gamma\omega}}{\rho} e^{-\gamma^2/2\alpha}.$$

As Rademacher observed in 1956 [**Ra**], the Riemann hypothesis plus Weyl's criterion imply that, for each $\omega > 0$, the sequence

$$\{\exp(i\gamma\omega) : \zeta(1/2 + i\gamma) = 0, \gamma > 0\}$$

is *equidistributed* in the unit circle. So we may expect a fair amount of cancellation to take place in the series $H(\omega)$. This may help to explain why it is so difficult to find a number $\omega$ for which $H(T, \omega)$ exceeds 1.

We reflect, for a moment, on the Weil explicit formula. This is an identity between two distributions [**Lang**, p. 339], [**Pa**, p. 39]. It is well established that certain classical explicit formulas follow from the

Weil explicit formula, by picking suitable test functions. For example, classical formulas for Dirichlet $L$-series may be derived in this way; see [**Lang**, Theorem 3.2, p. 340]. We are led to ask whether the Lehman formula can be obtained from the Weil explicit formula by picking a suitable test function.

After this speculation, we will simply state the theorem of Lehman.

### 6.2.1. Lehman's theorem.

**Theorem 6.4** (Lehman [**Leh**]). *Let $A$ be a positive number such that $\beta = \frac{1}{2}$ for all zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ for which $0 < \gamma \le A$. Let $\alpha$, $\eta$, and $\omega$ be positive numbers such that $\omega - \eta > 1$ and*

$$2/A \le 2A/\alpha \le \eta \le \omega/2.$$

*Let*

$$K(y) := \sqrt{\frac{\alpha}{2\pi}} e^{-\alpha y^2/2},$$

$$I(\omega, \eta) := \int_{\omega-\eta}^{\omega+\eta} K(u - \omega) u e^{-u/2} \{\pi(e^u) - \mathrm{li}(e^u)\} du.$$

*Then for $2\pi e < T \le A$*

$$I(\omega, \eta) = -1 - \sum_{0<|\gamma|\le T} \frac{e^{i\gamma\omega}}{\rho} e^{-\gamma^2/2\alpha} + R \tag{6.4}$$

*where $R$ admits an explicit bound with leading term*

$$\frac{3}{\omega - \eta} + 4(\omega + \eta) e^{-(\omega-\eta)/6}.$$

**6.2.2. Interpretation of Lehman's theorem.** We will consider the term

$$\frac{e^{i\gamma\omega}}{\rho} e^{-\gamma^2/2\alpha}$$

which appears in Lehman's theorem. For those $\rho$ which satisfy the Riemann hypothesis we can obtain this term by substituting $\gamma$ for $t$ in the following expression:

$$e^{-t^2/2\alpha} (1/2 + it)^{-1} e^{it\omega}.$$

Now the complex number $(1/2+it)^{-1}$ tends to 0 as $t \to \infty$ and stays in the sector

$$\{z : \Re(z) > 0, \Im(z) < 0\}$$

and therefore

$$\mathfrak{S} : t \mapsto e^{-t^2/2\alpha}(1/2+it)^{-1}e^{it\omega}$$

is a spiral curve, tending to 0 as $t \to \infty$.

If we replace $\rho = 1/2+i\gamma$ by the conjugate zeta zero $\overline{\rho} = 1/2-i\gamma$, then the term

$$\frac{e^{i\gamma\omega}}{\rho}e^{-\gamma^2/2\alpha}$$

is replaced by its complex conjugate. So the sum of this term and its conjugate is equal to

$$2\Re\left(\frac{e^{i\gamma\omega}}{\rho}\right)e^{-\gamma^2/2\alpha}.$$

We note that the complex numbers $e^{i\gamma\omega}$ are equidistributed on the unit circle. Now consider the countable set

$$\left\{\frac{e^{i\gamma\omega}}{\frac{1}{2}+i\gamma}e^{-\gamma^2/2\alpha} : \zeta\left(\frac{1}{2}+i\gamma\right) = 0\right\} \subset \mathfrak{S}.$$

Owing to the equidistribution phenomenon, it would seem difficult, a priori, for the real parts of these points to be sufficiently *negative* so that

$$I(\omega,\eta) > 0.$$

Littlewood's theorem guarantees that this phenomenon can indeed happen.

*The main conceptual breakthrough in Lehman's theorem is to replace the finitely many spirals in the Riemann formula by a single spiral. The main practical breakthrough is one of computability: the expression $I(\omega,\eta)$ is computable. For a large value of $T$, this will require a powerful computer.*

**6.2.3. A refinement of Lehman's theorem.** In the literature on the distribution of prime numbers, there is a maze of inequalities for $\pi(x)$.

We have selected the recent inequalities of Dusart [**D**, Theorem 5.1]. These important inequalities are very useful for us.

**Theorem 6.5.** *For $x \geq 4 \cdot 10^9$,*

$$\pi(x) \leq \frac{x}{\ln x}\left(1 + \frac{1}{\ln x} + \frac{2}{\ln^2 x} + \frac{7.32}{\ln^3 x}\right). \tag{6.5}$$

*For $x > 1$,*

$$\pi(x) \leq \frac{x}{\ln x}\left(1 + \frac{1}{\ln x} + \frac{2}{\ln^2 x} + \frac{7.59}{\ln^3 x}\right). \tag{6.6}$$

Since we are dealing with large values of $x$, we will be using the first inequality. The following result is a modest improvement on [**SD**], which is itself an improvement on [**CP**].

**Theorem 6.6.** *Under the hypothesis of Lehman's theorem and if $\omega - \eta > 44.22$, equation* (6.4) *still holds if the leading term in $R$ is replaced by*

$$\frac{2}{\omega-\eta} + \frac{8}{(\omega-\eta)^2} + \frac{58.56}{(\omega-\eta)^3} + \log 2 \cdot (\omega+\eta)e^{-(\omega-\eta)/2}$$
$$+ \frac{2}{\log 2}(\omega+\eta)e^{-(\omega-\eta)/6}.$$

**Proof.** We recall the definition

$$J(x) := \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots$$

and the Riemann formula

$$J(x) = \mathrm{Li}(x) - \sum_\rho \mathrm{Li}(x^\rho) + \int_x^\infty \frac{du}{(u^2-1)u\log u} - \log 2$$

valid for $x > 2$. We have

$$\frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots \leq \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3})\left\lfloor \frac{\log x}{\log 2} \right\rfloor.$$

Now we apply Dusart's inequality (6.5) to the term $\frac{1}{2}\pi(x^{1/2})$, and the classic bound $\pi(x) \leq \frac{2x}{\log x}$ to the term $\pi(x^{1/3})$. If $x \geq 4^2 \cdot 10^{18}$, then we have

$$\frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots \leq \frac{x^{1/2}}{\log x}\left(1 + \frac{2}{\log x} + \frac{8}{\log^2 x} + \frac{58.56}{\ln^3 x}\right) + 2\left\lfloor \frac{\log x}{\log 2} \right\rfloor \frac{x^{1/3}}{\log x}$$

and therefore

$$\frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots \leq \frac{x^{1/2}}{\log x}\left(1 + \frac{2}{\log x} + \frac{8}{\log^2 x} + \frac{58.56}{\ln^3 x}\right) + \frac{2}{\log 2} x^{1/3}.$$

Substituting in the Riemann formula, we have

$$\pi(x) \geq \mathrm{Li}(x) - \sum \mathrm{Li}(x^\rho) - \frac{x^{1/2}}{\log x}\left(1 + \frac{2}{\log x} + \frac{8}{\log^2 x} + \frac{58.56}{\ln^3 x}\right) - \frac{2}{\log 2} x^{1/3} - \log 2.$$

Now write $x = e^u$, and then if $u > 44.22$, we have

$$\pi(e^u) \geq \mathrm{Li}(e^u) - \sum \mathrm{Li}(e^{\rho u}) - \frac{e^{u/2}}{u}\left(1 + \frac{2}{u} + \frac{8}{u^2} + \frac{58.56}{u^3}\right) - \frac{2}{\log 2} e^{u/3} - \log 2.$$

Multiplying through by $ue^{-u/2}$, we obtain

$$ue^{-u/2}\left(\pi(e^u) - \mathrm{Li}(e^u)\right) \geq -1 - \sum_\rho ue^{-u/2}\mathrm{Li}(e^{\rho u}) - F(u) \tag{6.7}$$

where

$$F(u) := \frac{2}{u} + \frac{8}{u^2} + \frac{58.56}{u^3} + \frac{2u}{\log 2} e^{-u/6} + \log 2 \cdot ue^{-u/2}.$$

Recall that

$$K(y) = \sqrt{\frac{\alpha}{2\pi}} e^{-\alpha y^2/2}.$$

We now integrate (6.7) against the function $K$ between the limits $\omega - \eta$ and $\omega + \eta$:

$$\begin{aligned}
&\int_{\omega-\eta}^{\omega+\eta} K(u-\omega) u e^{-u/2} \{\pi(e^u) - \mathrm{Li}(e^u)\} du \\
&\geq \int_{\omega-\eta}^{\omega+\eta} K(u-\omega) \left( -1 - \sum_{\rho} u e^{-u/2} \mathrm{Li}(e^{\rho u}) - F(u) \right) du \\
\text{(S1)} \quad &= - \int_{\omega-\eta}^{\omega+\eta} K(u-\omega) F(u) du \\
\text{(S2)} \quad &- \int_{\omega-\eta}^{\omega+\eta} K(u-\omega) du \\
\text{(S3)} \quad &- \int_{\omega-\eta}^{\omega+\eta} K(u-\omega) u e^{-u/2} \left( \sum_{\rho} \mathrm{Li}(e^{\rho u}) \right) du.
\end{aligned}$$

Our new leading term comes from bounding the expression (S1), which is given by

$$- \int_{\omega-\eta}^{\omega+\eta} K(u-\omega) \left( \frac{2}{u} + \frac{8}{u^2} + \frac{58.56}{u^3} + \frac{2u}{\log 2} e^{-u/6} + \log 2 \cdot u e^{-u/2} \right) du.$$

In this integral, we exploit the fact that $\int_{-\infty}^{\infty} K(y) dy = 1$, and we infer that

$$\begin{aligned}
|(\mathrm{S1})| \leq &\frac{2}{\omega - \eta} + \frac{8}{(\omega-\eta)^2} + \frac{58.56}{(\omega-\eta)^3} \\
&+ \frac{2(\omega+\eta)}{\log 2} e^{-(\omega-\eta)/6} + \log 2 \cdot (\omega+\eta) e^{-(\omega-\eta)/2}. \qquad \square
\end{aligned}$$

*From the point of view of numerical analysis, the crucial improvement in Theorem* 6.6 *is in replacing the dominant term* $3/(\omega - \eta)$ *by* $2/(\omega - \eta)$.

Estimating (S2) is straightforward. The most laborious part of the proof comes up in estimating (S3), the sum involving the complex zeta zeros, some of which may not lie on the critical line $\Re(s) = 1/2$. Lehman does not assume any properties of these zeros other than the fact that if they exist, the magnitude of their imaginary components exceeds 14, which is approximately the value of the imaginary component of the first zeta zero above the real axis. The rest of the proof is a steadfast application of Cauchy's integral formula to the expression in the integral. The details of Lehman's proof will not be repeated here; they can of course be found in his article [**Leh**].

## Chapter 7

# The prime number race

## 7.1. On the logarithmic density

Before embarking on computation, we may ask how frequently the prime number theorem undercounts the primes. To do this, we need a suitable concept of *density*. We follow [**RS**, p. 174] and [**LMP**, p. 3744]. So, given a subset $\mathcal{M} \subseteq \{x \in \mathbb{R} : x \geq 2\}$, the *logarithmic density* of $\mathcal{M}$ is defined as usual as

$$\delta(\mathcal{M}) := \lim_{X\to\infty} \frac{1}{\log X} \int_{t\in\mathcal{M}\cap[2,X]} \frac{dt}{t}.$$

We are here concerned with the set

$$\Pi := \{x \geq 2 : \pi(x) > \mathrm{Li}(x)\}.$$

Then the logarithmic density $\delta(\Pi)$ is a measure of how frequently the prime number theorem undercounts the primes.

Recall the linear independence hypothesis (LI), which asserts that the sequence of numbers $\gamma_n$ such tha $\zeta(\frac{1}{2} + i\gamma_n) = 0$ is linearly independent over $\mathbb{Q}$. As usual, the Riemann hypothesis is denoted (RH).

An important development concerning the "race" between $\pi(x)$ and $\mathrm{Li}(x)$ was the paper of Rubinstein and Sarnak [**RS**]. They have shown [**RS**, p. 175], conditional on (RH) and (LI), that

$$\delta(\Pi) = \Delta \approx 2.6 \times 10^{-7}.$$

The conclusion is that, as measured by the logarithmic density, there is a definite bias towards $\pi(x) \leq \mathrm{Li}(x)$. This may help to explain why the Skewes number $\Xi$ is so large.

Since $\pi(x)$ counts primes, it is natural to consider the actual primes in the race between $\pi(x)$ and $\mathrm{Li}(x)$. What can be said about the set of primes $p$ for which $\pi(p) > \mathrm{Li}(p)$?

Lichtman, Martin, and Pomerance [**LMP**] defined the discrete logarithmic density of a set $\mathcal{M} \subset \mathbb{R}_{\geq 2}$ relative to the prime numbers as

$$\delta'(\mathcal{M}) := \lim_{x\to\infty} \frac{1}{\log\log x} \sum_{p\leq x,\, p\in\mathcal{M}} \frac{1}{p}$$

if the limit exists. They also introduced the modified limit

$$\delta^*(\mathcal{M}) := \lim_{x\to\infty} \frac{1}{\log x} \sum_{p\leq x,\, p\in\mathcal{M}} \frac{\log p}{p}$$

and showed that if the modified limit exists, then it is equal to $\delta'(\mathcal{M})$.

**Theorem 7.1** ([**LMP**]). *Let the set* $\Pi$ *and the number* $\Delta = \delta(\Pi)$ *be defined as above. Assuming* RH *and* LI, *the discrete logarithmic density of* $\Pi$ *relative to the primes is*

$$\delta^*(\Pi) = \Delta.$$

This result demands a great deal of reflection. If we may rephrase the authors' own reflections [**LMP**, p. 3744], we may say that there is no conspiracy for the primes to lie in the set of real numbers $\Pi$ any more or less often than expected.

As Rubinstein and Sarnak say in their article [**RS**, p. 175], although the initial segment in which $\pi(x)$ loses to $\mathrm{Li}(x)$ is extremely long, the probability that $\pi(x)$ beats $\mathrm{Li}(x)$, although small, is still palpable.

Beating $\mathrm{Li}(x)$ is the subject of the next section.

## 7.2. Upper bounds for the Skewes number

The implementation of Theorem 6.4 consists of first choosing the parameters $A, \alpha, \eta, \omega$ such that the conditions in Lehman's theorem, Theorem 6.4, are satisfied and then deciding how many actual zeros should be entered into the calculation. Then the sum

$$-\sum_{0<|\gamma|\leq T} \frac{e^{i\gamma\omega}}{\rho} e^{-\gamma^2/2\alpha}$$

is computed over a wide range of $\omega$ to find heuristic values of $x$ where the sum could be expected to exceed 1. After that, calculations with the estimates of the error term $R$ are done to prove that $\pi(x)$ exceeds $\mathrm{Li}(x)$ around this value.

Lehman's formula has the potential to derive a good upper bound for the Skewes number because numerical data concerning the zeta zeros may be fed into it.

There are many obstacles in refining Lehman's result—our imperfect knowledge of the influence of the four parameters and the lack of more precise knowledge of the zeta zeros, not to mention our sole dependence on Riemann's formula.

Where does one go from here? Littlewood's theorem has illustrated well enough the oscillatory nature of the difference $\pi(x) - \mathrm{Li}(x)$. A possible direction for improvement would be to continue to reduce the error term $R$. There is of course always the option of including more zeros in Lehman's formula, but this will most likely not yield a great deal of improvement because the dominant term in the error comes from the range of the interval we wish to consider.

In the 21st century, the first authors to refine Lehman's theorem were Chao-Plymen [**CP**], swiftly followed by Saouter-Demichel [**SD**]:

| Year | near $x$ | $N$ | by |
|---|---|---|---|
| 2000 | $1.39822 \times 10^{316}$ | $1 \times 10^{6}$ | Bays and Hudson |
| 2010 | $1.39801 \times 10^{316}$ | $1 \times 10^{7}$ | Chao and Plymen |
| 2010 | $1.3971667 \times 10^{316}$ | $2.2 \times 10^{7}$ | Saouter and Demichel |

where $N$ is the number of complex zeta zeros used.

We quote from Stoll and Demichel [**SDem**, p. 2382]: Our analysis conducted with up to 200 billion complex zeros gives computational evidence that a crossover may exist near $1.397162914 \times 10^{316}$.

At any rate, a powerful computer is required. Some of the issues in handling a large number of zeta zeros are described in [**CP**] and will not be repeated here.

In this context, the following elementary result is useful: if $x > 1$ and $y > 0$, then we have

$$\begin{aligned}\mathrm{Li}(x+y) - \mathrm{Li}(x) &= \int_x^{x+y} \frac{dt}{\log t} \\ &< \frac{y}{\log x}.\end{aligned}$$

With this result, we can state:

**Lemma 7.2.** *Let $x \geq 2$ be such that $\pi(x) - \mathrm{Li}(x) = M > 0$. Then if $y$ is a real number such that $0 < y < M \cdot \log x$, we have $\pi(x+y) - \mathrm{Li}(x+y) > 0$.*

**Proof.** Let $y > 0$. Since the function $\pi(x)$ is increasing, we then have

$$\begin{aligned}\pi(x+y) - \mathrm{Li}(x+y) &= (\pi(x+y) - \pi(x)) + (\pi(x) - \mathrm{Li}(x)) \\ &\quad + (\mathrm{Li}(x) - \mathrm{Li}(x+y)) \\ &> M - \frac{y}{\log x}.\end{aligned}$$ ☐

This shows that if $\pi(e^u) - \mathrm{Li}(e^u) = M > 0$, then $\pi(x) - \mathrm{Li}(x)$ will remain positive for another $[Mu]$ consecutive integers.

After a judicious choice of parameters and after computing the first 22 million zeta zeros, Saouter-Demichel [**SD**, Theorem 4.1] obtained the following result.

**Theorem 7.3.** *There exists at least one value $x$ in the interval*

$$[\exp 727.9513130, \exp 727.9513586]$$

*for which $\pi(x) > \mathrm{Li}(x)$. Moreover, there are more than $6.09 \times 10^{150}$ successive integers in the vicinity of $\exp 727.951335792$ where the inequality holds.*

In the same article [**SD**], the authors display the Bays-Hudson region, the Chao-Plymen region, and their own sharp region.

Stefanie Zegowitz [**Z**] has obtained a modest improvement on Theorem 7.3 with the interval

$$[\exp 727.951324783, \exp 727.951346801].$$

Stefanie's upper bound provides, at the time of writing , the best estimate for the Skewes number:

$$19 \log 10 < \log \Xi \leq 727.951346801.$$

We should also note that, *conditional on the Riemann hypothesis*, Chris Smith, in his MSc thesis, has obtained an improved estimate, `http://etheses.whiterose.ac.uk/id/eprint/16409`.

## Chapter 8

# Exercises, hints, and selected solutions

## 8.1. Exercises

1.1. Compute $\pi(10), \pi(20), \pi(30), \pi(40), \pi(50)$.

1.2. Write down a sequence of 333 consecutive nonprimes.

1.3. (Largest known prime) Estimate the length in kilometers of the largest known prime number (see `www.mersenne.org`).

1.4. Prove that

$$\zeta^2(s) = \sum \frac{d(n)}{n^s}$$

where $d(n)$ is the number of divisors of $n$.

1.5. Prove that

$$\zeta(s)\zeta(s-1) = \sum \frac{\sigma(n)}{n^s}$$

where $\sigma(n)$ denotes the sum of the divisors of $n$.

1.6. (An argument of Euler) What is wrong with the argument

$$\sum_{n=-\infty}^{+\infty} z^n = \frac{1}{1-z} + \frac{1/z}{1-1/z} = 0?$$

1.7. (Revision of poles) Let $f(s)$ have a simple pole at $s = a$. Prove that

$$(s-a)\frac{f'(s)}{f(s)} \longrightarrow -1$$

as $s \to a$.

1.8. A computer is programmed to add 1 trillion terms of the harmonic series $\sum \frac{1}{n}$ each second. Since the Big Bang, has enough time elapsed for the $N$th partial sum to exceed 100? [1 year $= 3.156 \times 10^7$ seconds.]

1.9. Suppose that we used the function $\mathrm{li}(x)$ instead of $\mathrm{Li}(x)$. In that case, what is the Skewes number relative to $\mathrm{li}(x)$, i.e., the least number for which $\pi(x) > \mathrm{li}(x)$?

1.10. (Life of Riemann) What was Riemann like? When did he live? What kind of life did he have?

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2.1. (Theorem of Euler, 1748) Prove that

$$\begin{aligned}1 - 1/2 - 1/3 - 1/5 + 1/6 - 1/7 + 1/10 - 1/11 - 1/13 + 1/14 + 1/15 \\ - 1/17 - 1/19 + 1/21 + 1/22 - 1/23 + 1/26 - 1/29 + \cdots = 0.\end{aligned}$$

2.2. The Möbius function $\mu(n)$ is defined by

$$\mu(1) = 1,\ \mu(p) = -1,\ \mu(p^m) = 0 \quad (m > 1),\ \mu(ab) = \mu(a)\mu(b)$$

with $(a, b) = 1$. Prove that for $\Re(s) > 1$ we have

$$\zeta(s)^{-1} = \sum \mu(n)/n^s.$$

2.3. (Identity of Ramanujan) Prove that

$$\frac{\zeta^2(s)}{\zeta(2s)} = \sum \frac{2^{\omega(n)}}{n^s}$$

where $\omega(n) = \#$ distinct prime factors of $n$.

2.4. (Revision) Define *Laurent series, residue*. What is the Laurent expansion of $\zeta(s)$ around $s = 1$? What is the residue of $\zeta(s)$ at $s = 1$?

2.5. (Ramanujan) What manner of man was Ramanujan? Why did he come to England? Where did his ideas come from?

2.6. (Millenium problems, including the Riemann hypothesis) Look up `www.claymath.org/millennium`.

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3.1. (Euler's constant $\gamma$) Let

$$t_n := 1 + 1/2 + 1/3 + \cdots + 1/(n-1) - \log n.$$

Prove that $t_n$ is an increasing sequence. Write

$$t_n = \sum_{k=1}^{n-1} \left( \frac{1}{k} - \log \frac{k+1}{k} \right)$$

and prove that $t_n$ is bounded above. Deduce that

$$1 + 1/2 + 1/3 + 1/4 + \cdots + 1/n - \log n \longrightarrow \gamma$$

as $n \to \infty$. Deduce also that

$$(1 + 1/2 + 1/3 + \cdots + 1/n)/\log n \to 1$$

as $n \to \infty$.

3.2. (Euler product again) Prove that

$$\frac{\zeta(s)}{\zeta(2s)} = \sum \frac{q(n)}{n^s}$$

where $q(n) = 1$ if $n$ is square-free and $q(n) = 0$ otherwise.

3.3. Let $d_k(n) := \#\{(x_1, \ldots, x_k) \in \mathbf{N}^k : x_1 \cdots x_k = n\}$. Prove that

$$\zeta(s)^k = \sum \frac{d_k(n)}{n^s}.$$

3.4. Look up Euler on `www.google.com`. What were his main contributions to pure mathematics?

3.5. Prove that

$$\int_0^1 \frac{x}{\log \frac{1}{1-x}} dx = \log 2.$$

3.6. Evaluate $\zeta(-5), \zeta(-7), \zeta(-9)$.

3.7. Find a formula for $\zeta(-2n-1)$.

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4.1. Sketch the step function

$$y = \sum_{n<x} \Lambda(n), \qquad 1 \leq x \leq 30.$$

4.2. Sketch the step function

$$y = \sum_{n \leq x} \Lambda(n), \qquad 1 \leq x \leq 30.$$

4.3. Sketch carefully the step function

$$y = \psi(x), \qquad 1 \leq x \leq 30.$$

What do you notice about this graph?

4.4. Show that

$$\frac{x^\rho}{\rho} + \frac{x^{\overline{\rho}}}{\overline{\rho}} = \frac{2x^\beta}{\beta^2 + \gamma^2}(\beta \cos(\gamma \log x) + \gamma \sin(\gamma \log x))$$

where $\rho = \beta + i\gamma$.

4.5. Recall from Chapter 4 the function

$$J(x) = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots$$

which features in the Riemann paper [**R**]. Show that

$$J(x) \sim \mathrm{Li}(x)$$

as $x \to \infty$.

---

5.1. (Mellin transform) The Mellin transform is

$$\mathcal{M}f(s) := \int_1^\infty f(x)x^{-s-1}dx.$$

Prove that $\mathcal{M}(1)(s) = 1/s$ if $\sigma > 0$.

5.2. Compute the Mellin transform of $x$.

5.3. Prove that

$$\mathcal{M}(x^\rho)(s) = 1/(s-\rho).$$

5.4. Prove that

$$\mathcal{M}(\log(1-1/x^2))(s) = -\sum_{n=1}^{\infty} 1/2n(s+2n).$$

5.5. In the formula

$$\pi(x) = \sum \frac{\mu(r)}{r} J(x^{1/r})$$

Riemann replaced $J(x)$ by $\mathrm{Li}(x)$ to get the approximate formula

$$\pi(x) \approx \sum \frac{\mu(r)}{r} \mathrm{Li}(x^{1/r}).$$

Prove the Ramanujan approximate formula

$$\frac{d\pi(x)}{dx} \approx \frac{1}{x\ln x} \sum \frac{\mu(r)}{r}(x^{1/r}).$$

5.6. (Ramanujan identity) Prove that

$$\frac{\zeta^3(s)}{\zeta(2s)} = \sum \frac{d(n^2)}{n^s}$$

where $d(n)$ is as in Exercise 1.4.

*Ramanujan*, 1887 –1920. Indian genius: See *The Man Who Knew Infinity* by Robert Kanigel, Abacus, paperback. See also the novel *The Indian Clerk* by David Leavitt.

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6.1. (L'Hôpital's rule) Prove the asymptotic formula

$$\mathrm{Li}(x) \sim \frac{x}{\log x}$$

as $x \to \infty$.

6.2. (Ramanujan identity) Prove that

$$\frac{\zeta^3(s)}{\zeta(2s)} = \sum \frac{d(n^2)}{n^s}$$

where $d(n)$ is as in Exercise 1.4.

6.3. (Ramanujan identity) Prove that

$$\frac{\zeta^4(s)}{\zeta(2s)} = \sum \frac{d(n)^2}{n^s}.$$

6.4. Sketch the graphs of the first three oscillatory terms in the von Mangoldt formula

$$\psi(x) = x - \sum x^\rho/\rho + O(1).$$

6.5. Is the von Mangoldt formula a Fourier expansion? If not, why not?

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7.1. Prove that the Fourier series for $f(x) = x$ $(-\pi \le x \le \pi)$ is given by

$$x = 2\left(\frac{1}{1}\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right)$$

and sketch the right-hand side.

Taking $x = \pi/2$, obtain the Leibnitz formula:

$$\pi = \frac{4}{1} - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots .$$

7.2. Prove that, for $0 \leq x \leq 1$,

$$x - \frac{1}{2} = -2 \sum_{n \geq 1} \frac{\sin(2\pi n x)}{2\pi n}.$$

Compare this with the von Mangoldt formula.

7.3. Define a topology on $\mathbb{Z}$ as follows: $E$ is open if $E = \emptyset$ or $E$ is a union of arithmetic progressions. Prove that an arithmetic progression is open and closed. Define

$$A = \bigcup A_p$$

where $A_p = \{n \in \mathbb{Z} : n \equiv 0 \mod p\}$. Prove that $\mathbb{Z} - A = \{-1, 1\}$ is not open and $A$ is not closed. Deduce that there are infinitely many primes.

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8.1. [Revision] Define *Taylor series, Laurent series, isolated singularity, isolated essential singularity, pole of order n, simple pole, residue, zero of order n, simple zero, holomorphic function in an open set, meromorphic function, meromorphic continuation.*

Define the function $\Gamma(s)$ and write down its basic properties. Where are the poles of $\Gamma(s)$? What are the residues at these poles? Where are the zeros of $\Gamma(s)$?

8.2. The Riemann-Siegel function $\Xi(t)$ is defined by

$$\Xi(t) = -\frac{1}{2}\left(t^2 + \frac{1}{4}\right) \pi^{-\frac{1}{4} - \frac{it}{2}} \, \Gamma\left(\frac{1}{4} + \frac{it}{2}\right) \zeta\left(\frac{1}{2} + it\right).$$

This function is entire, real valued for real $t$, and even: $\Xi(t) = \Xi(-t)$. How would you use this function to locate the zeros of $\zeta(s)$ on $\Re(s) = \frac{1}{2}$?

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## 8.2. Hints and selected solutions

Notation: $O(1)$ means a bounded function, $\Re(s)$ is the real part of $s$, and $\log x$ and $\ln x$ both mean the natural log.

1.1. $\pi(10) = \#\{2, 3, 5, 7\} = 4$, $\pi(20) = \pi(10) + \#\{11, 13, 17, 19\} = 8$, $\pi(30) = \pi(20) + \#\{23, 29\} = 10$, $\pi(40) = \pi(30) + \#\{31, 37\} = 12$, $\pi((5) = \pi(40) + \#\{41, 43, 47\} = 15$.

1.2. $334! + 2, 334! + 3, \ldots, 334! + 334$.

1.3. The largest known prime has $24,862,048$ digits, see `www.mersenne.org`. At $1,000$ digits/meter, this has length $24,862$ meters $\approx 25$ kilometers.

1.4. We have

$$\begin{aligned}
\zeta^2(s) &= \sum \frac{1}{m^s} \cdot \sum \frac{1}{n^s} \\
&= \sum_{m,n} \frac{1}{(mn)^s} \\
&= \sum \frac{\overbrace{1 + \cdots + 1}}{N^s} \\
&= \sum \frac{d(N)}{N^s}
\end{aligned}$$

since the number of 1's is precisely $d(N)$.

1.5. We have

$$\begin{aligned}
\zeta(s)\zeta(s-1) &= \sum \frac{1}{m^s} \cdot \sum \frac{n}{n^s} \\
&= \sum_{m,n} \frac{n}{(mn)^s} \\
&= \sum \frac{\sigma(N)}{N^s}
\end{aligned}$$

since, for a fixed $N$, the numbers in the numerator are precisely the divisors of $N$.

1.6. The series $\sum z^n$ is made of two series as follows:

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \qquad |z| < 1,$$

$$\sum_{n=1}^{\infty} 1/z^n = \frac{1/z}{1-1/z}, \qquad |1/z| < 1,$$

and there is *no* $z$ which satisfies both inequalities.

1.7. We have $f(s) = (s-a)^{-1}g(s)$ with $g$ holomorphic and $g(a) \neq 0$. Then

$$\begin{aligned}(s-a)\frac{f'(s)}{f(s)} &= (s-a) \cdot \frac{(s-a)^{-1}g'(s) - (s-a)^{-2}g(s)}{(s-a)^{-1}g(s)} \\ &= (s-a)\frac{g'(s)}{g(s)} - 1.\end{aligned}$$

This tends to $-1$ as $s \to a$.

1.8. The number of terms is

$$15 \times 10^9 \times 3.156 \times 10^7 \times 10^{12} \approx 4.5 \times 10^{29} \approx 4.5 \times e^{2.3\times 29} \approx e^{66.7}.$$

But

$$\sum_{n=1}^{N} \frac{1}{n} \approx \log N$$

and so $N \approx e^{100}$. Not enough time has elapsed.

1.9. Here we have $\pi(2) = 1, \mathrm{li}(2) = 0$. The Skewes number relative to $\mathrm{li}(x)$ is therefore equal to 2. This would, as one reviewer wryly observed, spoil the fun!

1.10. Go to

`www-history.mcs.st-andrews.ac.uk/Mathematicians/Riemann.html`.

2.1. Note the pattern. Only square-free numbers appear. Each prime has $-1$ attached to it. Products of 2 primes have $+1$.

Now the Euler product formula for $\zeta(s)$ implies that for $\Re(s) > 1$

$$\begin{aligned}\frac{1}{\zeta(s)} &= \prod_p (1 - p^{-s}) \\ &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.\end{aligned}$$

Since $\zeta(s)$ has a simple pole at $s = 1$, $[\zeta(s)]^{-1}$ has a simple zero at $s = 1$. So if the above equation were valid for $s = 1$, it would say

$$0 = \sum_{n=1}^{\infty} \frac{\mu(n)}{n}$$

as required. For a further discussion of this formula, see [**Ed**, p. 92]: von Mangoldt provided a rigorous proof in 1897.

2.2. The Euler product formula gives

$$\zeta(s)^{-1} = \prod (1 - p^{-s}) = (1 - 2^{-s})(1 - 3^{-s})(1 - 5^{-s}) \cdots .$$

Now multiply out this infinite product and note that the pattern formation is the same as in the definition of $\mu$.

2.3. The Euler product gives

$$\begin{aligned}\frac{\zeta^2(s)}{\zeta(2s)} &= \prod \frac{(1 - p^{-s})^{-2}}{(1 - p^{-2s})^{-1}} \\ &= \prod \frac{(1 - p^{-2s})}{(1 - p^{-s})^2} \\ &= \prod \frac{1 + p^{-s}}{(1 - p^{-s})}.\end{aligned}$$

This expression is

$$\prod (1+p^{-s})(1+p^{-s}+p^{-2s}+\cdots) = \prod (1+2p^{-s}+2p^{-2s}+2p^{-3s}+\cdots).$$

Now multiply out this infinite product. Each term in the resulting sum is of the form

$$\frac{2 \times 2 \times \cdots \times 2}{p_1^{k_1 s} p_2^{k_2 s} \cdots p_r^{k_r s}} = \frac{2^{\omega(N)}}{N^s}$$

since $r = \omega(N)$.

2.4. The answers are all in Chapter 1.

2.5. There are several excellent sources for the life and work of Ramanujan. For a detailed account, go to the famous book of G. H. Hardy [**H**]. For a briefer but very incisive account, see the chapter on Ramanujan in the book by Stephen Wolfram [**W**]. Finally, you may go to

`www-history.mcs.st-andrews.ac.uk/Mathematicians/Ramanujan.html`.

2.6. Go to `www.claymath.org/millennium`.

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3.1. Sketch the graph of $y = 1/x$ and recall that

$$\ln(n) = \int_1^n \frac{1}{x}.$$

Mark the points $x = 1, 2, 3, \ldots, n-1, n$. Now compare the area under the graph with the area under the obvious rectangles. It is then clear that $t_n$ is an increasing sequence. We have

$$\begin{aligned} t_n &= \sum_{k=1}^{n-1} \left( \frac{1}{k} - \ln\left(1 + \frac{1}{k}\right)\right) \\ &= \sum_{k=1}^{n-1} \frac{1}{2k^2} - \frac{1}{3k^3} + \frac{1}{4k^4} - \cdots \\ &< \sum_{k=1}^{n-1} \frac{1}{2k^2} \\ &< \frac{\pi^2}{12}. \end{aligned}$$

The sequence $t_n$ is increasing and bounded above and therefore has a limit, denoted $\gamma$. Note that $1/(n-1)$ is a null sequence. Since $t_n \to \gamma$, we have $t_n/\ln(n) \to 0$ as required.

3.2. We have

$$\begin{aligned}\frac{\zeta(s)}{\zeta(2s)} &= \prod \frac{(1-p^{-s})^{-1}}{(1-p^{-2s})^{-1}} \\ &= \prod \frac{(1-p^{-2s})}{(1-p^{-s})} \\ &= \prod (1+p^{-s}).\end{aligned}$$

This infinite product is

$$(1+2^{-s})(1+3^{-s})(1+5^{-s})\cdots .$$

Now multiply out. In the resulting sum, only *square-free* denominators appear, with coefficient 1. This coincides with the definition of $q(n)$.

3.3. We have

$$\begin{aligned}\zeta^k(s) &= \sum \frac{1}{n_1^s} \cdots \sum \frac{1}{n_k^s} \\ &= \sum \frac{\overbrace{1+1+\cdots+1}}{n_1^s \cdots n_k^s} \\ &= \sum \frac{d_k(N)}{N^s}\end{aligned}$$

since the number of 1's is precisely $d_k(N)$.

3.4. Go to

`www-history.mcs.st-andrews.ac.uk/Mathematicians/Euler.html`.

3.5. Make the substitution $x \mapsto 2x - x^2$ in the integral

$$\int_{2\epsilon-\epsilon^2}^{1} (\log(1-x))^{-1} dx.$$

The significance of this example is that the integral of a transcendental function can be a period "by accident"; see [**KZ**, §1.1].

3.6. Use the functional equation.

3.7. Use the functional equation and the Euler formula for $\zeta(2n)$.

4.1.–4.3. Each graph is a step function with jump discontinuities at each prime and prime power, i.e., at

$$x = 2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 23, 25, 27, 29.$$

The jumps are

$$\ln 2, \ln 3, \ln 2, \ln 5, \ln 7, \ln 2, \ln 11, \ln 13, \ln 2, \ln 17, \ln 23, \ln 5, \ln 3, \ln 29.$$

Note that, in (4.2), the value at $x = 7$ is $\ln 2 + \ln 3 + \ln 2 + \ln 5 + \ln 7$; in (4.1), the value at $x = 7$ is $\ln 2 + \ln 3 + \ln 2 + \ln 5$.

In (4.3), the value at $x = 7$ is $\ln 2 + \ln 3 + \ln 2 + \ln 5 + \frac{1}{2} \ln 7$, the *average* of the values in (4.1) and(4.2).

In (4.1) each step is an interval of the form $(a, b]$. In (4.2) each step is of the from $[a, b)$. In (4.3) each step is an open interval of the form $(a, b)$.

If your graph is accurate, you will notice that

$$\psi(x) \sim x;$$

i.e., $\psi(x)$ will look like, from a distance, the straight line $y = x$.

Since the slope of $\psi(x)$ is roughly $\pi/4$, the graph is trying to tell you that the average space between primes around $p$ is $\log p$. This is the first intimation of the prime number theorem.

4.4. The left-hand side is

$$\begin{aligned} 2\Re \frac{x^\rho}{\rho} &= 2\Re \frac{\overline{\rho}}{|\rho|^2} e^{\rho \ln x} \\ &= \frac{2}{\gamma^2 + \beta^2} \Re \left( \overline{\rho} e^{\rho \ln x} \right) \\ &= \frac{2x^\beta}{\gamma^2 + \beta^2} \Re (\beta - i\gamma) e^{i\gamma \ln x} \end{aligned}$$

and we also have

$$\Re (\beta - i\gamma) e^{i\gamma \ln x} = \beta \cos(\gamma \ln x) + \gamma \sin(\gamma \ln x).$$

4.5. Start with the fact that

$$J(x)-\pi(x)=\frac{1}{2}\pi(x^{1/2})+\cdots+\frac{1}{n}\pi(x^{1/k}),$$

a finite series. Since $\pi(x)<x$, we get

$$\begin{aligned} J(x)-\pi(x) &< \frac{1}{2}x^{1/2}+\cdots+\frac{1}{n}x^{1/k} \\ &< \frac{1}{2}x^{1/2}+\cdots+\frac{1}{2}x^{1/2} \\ &= (k-1)\cdot\frac{x^{1/2}}{2}. \end{aligned}$$

But $k$ is the integer part of $\frac{\log x}{\log 2}$ so we get

$$J(x)-\pi(x)=O(x^{1/2}\log x).$$

Then we have

$$\frac{J(x)}{x/\log x}=\frac{\pi(x)}{x/\log x}+O\left(\frac{(\log x)^2}{\sqrt{x}}\right).$$

Now let $x\to\infty$ and use the prime number theorem.

---

5.1. Let $a=0$ in the solution to Exercise 5.3.

5.2. Let $a=1$ in the solution to Exercise 5.3. We get

$$\mathcal{M}(x)(s)=\frac{1}{s-1}$$

if $\Re(s)>1$.

5.3. We have

$$\begin{aligned} \mathcal{M}(x^a) &= \int_1^\infty x^{a-s-1}dx \\ &= \left[\frac{x^{a-s}}{a-s}\right]_1^\infty \\ &= \frac{1}{s-a} \end{aligned}$$

if $\Re(s)>\Re(a)$. Now let $a=\rho$, a complex zero of $\zeta(s)$. If the Riemann hypothesis is true, then the condition is $\Re(s)>1/2$.

5.4. We have

$$\begin{aligned}\mathcal{M}\log(1-1/x^2)(s) &= -\mathcal{M}\sum\frac{x^{-2n}}{n}(s)\\ &= -\sum\frac{1}{n}\mathcal{M}x^{-2n}(s)\\ &= -\sum\frac{1}{n(s+2n)}\end{aligned}$$

by the solution to Exercise 5.3 with $a=-2n$, if $\Re(s)>-2$.

5.5. All you have to do is apply the chain rule to $\mathrm{Li}(x^{1/r})$. This gives

$$\begin{aligned}\frac{d}{dx}\mathrm{Li}(x^{1/r}) &= \frac{1}{\ln x^{1/r}}\cdot(1/r)x^{1/r-1}\\ &= \frac{1}{x\ln x}\cdot x^{1/r}.\end{aligned}$$

The graph of $y=\pi(x)$ is a step function, with a jump discontinuity at each prime $p$. If you look at this graph from a great distance, you will see a smooth curve, and the Ramanujan formula tells you the *slope* of this curve.

5.6. See the solution to Exercsie 6.3.

---

6.1. As $x\to\infty$ we have

$$\begin{aligned}\lim\frac{\mathrm{Li}(x)}{x/\ln x} &= \lim\frac{(\ln x)^{-1}}{(\ln x)^{-1}+(\ln x)^{-2}}\\ &= \lim\frac{1}{1+(\ln x)^{-1}}\\ &= 1.\end{aligned}$$

6.2. We have the identity

$$\begin{aligned}\frac{1-x^2}{(1-x)^3} &= \frac{1+x}{(1-x)^2}\\ &= (1+x)(1+2x+3x^2+4x^3+\cdots)\\ &= 1+3x+5x^2+7x^3+\cdots.\end{aligned}$$

It follows that

$$\begin{aligned}\frac{\zeta^3(s)}{\zeta(2s)} &= \prod \frac{(1-p^{-s})^{-3}}{(1-p^{-2s})^{-1}}\\ &= \prod \frac{(1-p^{-2s})}{(1-p^{-s})^3}\\ &= \prod \sum (2k+1)p^{-ks}.\end{aligned}$$

Note that $d(p^{2k}) = \#\{1, p, p^2, \ldots, p^{2k}\} = 2k+1$. Note also that $d$ is multiplicative; i.e., $d(ab) = d(a)d(b)$ when $(a,b) = 1$. The last term is

$$\begin{aligned}\prod \sum d(p^{2k})p^{-ks} &= \sum \frac{d(p_1^{2k_1}) \cdots d(p_r^{2k_r})}{p_1^{k_1 s} \cdots p_r^{k_r s}}\\ &= \sum \frac{d(n^2)}{n^s}.\end{aligned}$$

6.3. We have the identity

$$\begin{aligned}\frac{1-x^2}{(1-x)^4} &= \frac{1+x}{(1-x)^3}\\ &= (1+3x+5x^2+7x^3+\cdots)(1+x+x^2+\cdots)\\ &= \sum (k+1)^2 x^k.\end{aligned}$$

It follows that

$$\begin{aligned}\frac{\zeta^4(s)}{\zeta(2s)} &= \prod \frac{(1-p^{-s})^{-4}}{(1-p^{-2s})^{-1}}\\ &= \prod \frac{(1-p^{-2s})}{(1-p^{-s})^4}\\ &= \prod \sum (k+1)^2 p^{-ks}.\end{aligned}$$

Note that $d(p^k) = \#\{1, p, p^2, \ldots, p^k\} = k+1$. The last term is

$$\begin{aligned}\prod \sum d(p^k)^2 p^{-ks} &= \sum \frac{d(p_1^{k_1})^2 \cdots d(p_r^{k_r})^2}{p_1^{k_1 s} \cdots p_r^{k_r s}}\\ &= \sum \frac{d(n)^2}{n^s}.\end{aligned}$$

Examples of this kind may be found in Chapter I of the well-known book by Titchmarsh [**Ti**].

6.4. If the Riemann hypothesis is true, then the envelope of the graph for the complex zero $\rho$ is $y = 2\sqrt{x}/|\rho|$. This curve is the parabola

$$y^2 = \frac{4x}{|\rho|^2}.$$

As we move up the critical line $\Re(s) = 1/2$, these parabolas get closer and closer to the $x$-axis, and the oscillations get faster and faster.

6.5. The following formulation of the von Mangoldt formula, with $t = \log x$, has already been proved:

$$\frac{\Psi(x) - x}{\sqrt{x}} + O(1) = -2\sum_{n=1}^{\infty} \frac{\sin(\gamma_n t)}{\gamma_n}.$$

The *fundamental frequency* is $\gamma_1 = 14.134725\ldots$. The higher frequencies $\gamma_2, \gamma_3, \ldots$ are *not* integer multiples of $\gamma_1$ and so this is *not* a Fourier series, but we do have the same *idea* as in Fourier series, namely building up a function by superposition of waves.

---

7.1. We seek the Fourier series of the function $f(x)$, of period $2\pi$, for which $f(x) = x$ in $(-\pi, \pi)$. Now $f(x)$ is odd; therefore, $A_n = 0$. We have

$$\begin{aligned} B_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx dx \\ &= \frac{1}{\pi}\left[-\frac{x}{n}\cos nx\right]_{-\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx dx \\ &= -\frac{2}{n}\cos n\pi \end{aligned}$$

so we have

$$B_n = (-1)^{n+1}\frac{2}{n}.$$

For the graphs, see

`http://en.wikipedia.org/wiki/Fourier_series`.

7.2. We seek the Fourier series of the function $g(x)$, of period 1, for which $g(x) = x$ in $(0, 1)$, the Sawtooth wave. We have

$$A_0 = \int_0^1 x dx = 1/2$$

and, when $n \neq 0$,

$$\begin{aligned} A_n &= 2\int_0^1 x\cos(2\pi nx)dx \\ &= 2\left[x\frac{\sin 2\pi nx}{2\pi n}\right]_0^1 - 2\int_0^1 \frac{\sin(2\pi nx)}{2\pi n}dx \\ &= 0, \end{aligned}$$

$$\begin{aligned} B_n &= 2\int_0^1 x\sin(2\pi nx)dx \\ &= 2\left[-x\frac{\cos(2\pi nx)}{2\pi n}\right]_0^1 + 2\int_0^1 \frac{\cos(2\pi nx)}{2\pi n}dx \\ &= -\frac{1}{n\pi} \end{aligned}$$

and so we have

$$x = 1/2 - 2\sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{2\pi n}.$$

7.3. Each nonempty open set is a union of arithmetic progressions and is therefore countable. Therefore $\{-1, 1\}$, which is the complement of $A$, is not open. So $A$ is not closed. Now each $A_p$ is the complement of the union of finitely many arithmetic progression, and is therefore closed. If there are finitely many primes, then $A = \bigcup A_p$ is closed, contradiction.

---

8.1. The gamma function $\Gamma(s)$ is defined by the equation

$$\Gamma(s) := \frac{e^{-\gamma s}}{s\phi(s)}$$

where

$$\phi(s) = \prod(1 + s/n)e^{-s/n}.$$

$\Gamma(s)$ has simple poles at $k = 0, -1, -2, -3, \ldots$ with residues

$$\frac{(-1)^k}{k!}.$$

It satisfies the relation

$$\Gamma(s) = \frac{\Gamma(s+1)}{s}$$

and $\Gamma(n) = (n-1)!$. The gamma function is zero-free; i.e., it never takes the value 0.

8.2. In the formula for $\Xi(t)$ the terms apart from the zeta function are zero-free. So $\Xi(t) = 0$ iff $\zeta(1/2 + it) = 0$. Now $\Xi(t)$ is real valued and continuous, so the zeros on $\Re(s) = 1/2$ can be located by finding where $\Xi(t)$ crosses the axis, i.e., where $\Xi(t)$ *changes sign.* This is, in fact, how the complex zeros of $\zeta(s)$ are found.

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